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DETERMINANTS

Properties & Applications

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DETERMINANTS: DEFINITION

Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

The *determinant* of *A* is defined by

$$\det A \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \ . \tag{1}$$

Next, consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} .$$

To evaluate its determinant, we work as follows: First, we draw a 3×3 "chessboard" consisting of + (plus) and – (minus) signs, as shown below. *Careful:* At the *top left* we always put a *plus* sign!

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} .$$

We may now develop the determinant of *A with respect to any row or any column*; the result will always be the same. Let us assume, e.g., that we choose to develop with respect to the *first row*. Its first element is *a*. At the position where this element is located (top left) the "chessboard" has a + sign; we thus leave the sign of *a* unchanged. Imagine now that we *cross off both the row and the column to which this element belongs* (first row, first column in this case). What is left over is a lower-order, 2×2 matrix with determinant

$$\begin{vmatrix} e & f \\ h & k \end{vmatrix} .$$

We multiply this determinant by *a* and we save the result.

The second element in the first row is *b*. At its location, the chessboard has a – sign; we thus write –*b*. We "cross off" the row and the column to which *b* belongs (first row, second column) and we get the 2×2 determinant

$$\begin{vmatrix} d & f \\ g & k \end{vmatrix} .$$

We multiply this by –*b* and we save this result, too.

The third element in the first row is *c*. At its location the chessboard has a + sign, thus we leave the sign of *c* unchanged. Crossing off the first row and the third column, where *c* is located, we find the determinant

$$\begin{vmatrix} d & e \\ g & h \end{vmatrix} .$$

We multiply this by *c* and again we save this result in the "memory".

Summing the contents of the memory, we finally find the determinant of *A*:

$$\det A \equiv \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} . \qquad (2)$$

Of course, to complete the job we must evaluate the minor determinants according to Eq. (1), which is an easy task.

Exercise: Evaluate again the determinant of *A*, this time by developing with respect to the *second column*, and show that

$$\det A \equiv \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + e\begin{vmatrix} a & c \\ g & k \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} .$$

Verify that your result is the same as before.

Exercise: With the aid of the chessboard

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

(the + sign always on the *top left*!) and by following an analogous procedure, verify formula (1) for a 2×2 determinant. (By definition, the determinant of a 1×1 matrix [*a*] is equal to the single element of the matrix.)

For a 4×4 matrix, the chessboard is of the form (with a + sign always on the top left)

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix} .$$

The development of a 4×4 determinant leads to 3×3 determinants that are developed as shown previously. As is obvious, the problem becomes harder as the dimension of the determinant increases!

Exercise: Show that

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & -1 \end{vmatrix} = 0 ,$$

by developing with respect to a row and, again, with respect to a column. Choose the row and column that will make your calculations easier. (Obviously, as a general rule, it is in our best interest to choose a row or a column with *as many zeros as possible*!)

PROPERTIES OF DETERMINANTS

Let A be an $n \times n$ matrix and let $\det A$ be the determinant of A. The following statements are true:

1. If all elements of a row or a column of A are zero, then $\det A = 0$.

2. If every element of a row or a column of A is multiplied by λ, then $\det A$ is multiplied by λ as well.

3. If *all* elements of A are multiplied by λ, then $\det A$ is multiplied by λ^n (where n is the dimension of A). That is,

$$\det(\lambda A) = \lambda^n \det A .$$

4. If any two rows or any two columns of A are interchanged, the value of $\det A$ is multiplied by (–1).

5. If two rows or two columns of A are identical, then $\det A = 0$. The same is true, more generally, if two rows or two columns are multiples of each other.

6. The value of det$A$ remains the same if the rows and columns of A are interchanged. That is,

$$\det(A^T) = \det A \ ,$$

where A^T is the *transpose* of A: $(A^T)_{ij} = A_{ji}$.

7. If A and B are $n\times n$ matrices,

$$\det(AB) = \det(BA) = \det A \,.\, \det B \ .$$

Also,

$$\det(A^k) = (\det A)^k \ , \quad k=1,2,3,\ldots \ .$$

8. If A^{-1} is the *inverse* of A (see below),

$$\det(A^{-1}) = 1/\det A \ .$$

9. The determinant of a *diagonal* (or, more generally, a *triangular*) matrix A is equal to the product of the elements of the diagonal of A.

10. The value of det$A$ is unchanged if to any row or any column of A we add an arbitrary multiple of any other row or column, respectively.

EVALUATION OF A MATRIX INVERSE

Consider a 3×3 matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv [a_{ij}] \qquad (i, j = 1,2,3) \ .$$

Let a_{ij} be an arbitrary element of A (the one that belongs to the i-th row and the j-th column). By "crossing off" the row and the column to which a_{ij} belongs, we obtain a 2×2 matrix. We call D_{ij} the determinant of this latter matrix.

We now construct a 3×3 matrix M, as follows: We replace every element a_{ij} of the given matrix A by the corresponding quantity

$$(-1)^{i+j} D_{ij} \ .$$

That is, in place of a_{ij} we put the minor determinant D_{ij} *multiplied by the sign that exists on the chessboard at the position of* a_{ij} . We thus get

$$M = \begin{bmatrix} D_{11} & -D_{12} & D_{13} \\ -D_{21} & D_{22} & -D_{23} \\ D_{31} & -D_{32} & D_{33} \end{bmatrix} .$$

Finally, we take the *transpose* M^T of M, which is called the *adjoint* of the matrix A:

$$\text{adj}\, A = M^T = \begin{bmatrix} D_{11} & -D_{21} & D_{31} \\ -D_{12} & D_{22} & -D_{32} \\ D_{13} & -D_{23} & D_{33} \end{bmatrix} .$$

The *inverse* A^{-1} of A, satisfying $AA^{-1} = A^{-1}A = I$ (where I is the 3×3 unit matrix) is given by

$$A^{-1} = \frac{1}{\det A}\ \text{adj}\, A \qquad (3)$$

Obviously, a necessary condition in order that the inverse of A may exist (i.e., in order that the matrix A be *invertible*) is that $\det A \neq 0$. The process described above, leading to relation (3), is generally valid for *any* $n\times n$ matrix (n=2,3,4,...).

Exercise: For the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

show that

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

Verify that

$$AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Exercise: By using (3), show that

$$\begin{bmatrix} 0 & 1 & -3 \\ -1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ -1/2 & 0 & 3/2 \\ -1/2 & 0 & 1/2 \end{bmatrix} .$$

Verify that your result satisfies the relation $AA^{-1} = A^{-1}A = I$.

SOLUTION OF LINEAR SYSTEMS

The method we will describe applies to any *linear system of equations*; i.e., system of n linear equations with n unknowns (n=2,3,4,...). For simplicity, we consider a system of two equations:

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 &= b_2 \end{aligned} \qquad (4)$$

In matrix form, this is written

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad (5)$$

where A is the matrix of the coefficients of the unknowns, $\mathbf{x}$ is the column vector of the unknowns and $\mathbf{b}$ is the column vector of the constants. In the case where $\mathbf{b}=0 \Leftrightarrow b_1=b_2=0$, the given system is said to be *homogeneous linear*.

We note the following:

1. If $\det A \neq 0$, the matrix A is *invertible* and the system has a *unique solution* that is obtained as follows:

$$A\mathbf{x} = \mathbf{b} \;\Rightarrow\; A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \;\Rightarrow\; (A^{-1}A)\,\mathbf{x} = A^{-1}\mathbf{b} \;\Rightarrow$$

$$\mathbf{x} = A^{-1}\mathbf{b} \qquad (6)$$

In the case where $\mathbf{b}=0$ (homogeneous system), the only solution of the system is the trivial one: $\mathbf{x}=0 \Leftrightarrow x_1=x_2=0$.

2. If $\det A=0$ (the matrix A is *non-invertible*), the system either has no solution (is *inconsistent*) or has an *infinite number* of solutions (see below).

The difficulty in solving (6) lies in the necessity of determining the inverse matrix. Let us now see an alternative expression for the solution of the system, based on *Cramer's method* (or *method of determinants*). As before, we call A the matrix of the coefficients of the unknowns in system (4):

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

Furthermore, we call A_1 the matrix obtained from A by replacement of its first column (i.e., the column of the coefficients a_{11} and a_{21} of x_1) with the column of the constant terms b_1 and b_2 . Similarly, we call A_2 the matrix obtained from A by replacing its second column (the one with the coefficients of x_2) with the column of the constants. Analytically,

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix} \;,\quad A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix} .$$

Then, the solution of system (4) – if it exists – is written

$$x_1 = \frac{\det A_1}{\det A} \;,\quad x_2 = \frac{\det A_2}{\det A} \tag{7}$$

The determinants of the matrices A_1 and A_2 are called *Cramer's determinants*.

Exercise: Write the analytical expression of the general solution (7), for any given a_{ij} and b_i .

Exercise: Consider the system

$$\begin{gathered} a\,x + b\,y = c \\ e\,x + f\,y = g \end{gathered}$$

(where we have put x_1=x, x_2=y) . Show that its solution is

$$x = \frac{c\,f - bg}{a\,f - be} \;,\quad y = \frac{ag - c\,e}{a\,f - be} \;.$$

Assume now that we "rewrite" the system by inverting the order of the two equations:

$$\begin{gathered} e\,x + f\,y = g \\ a\,x + b\,y = c \end{gathered}$$

Must we expect a different solution? How is your answer related to the properties of determinants?

More generally, for a linear system of n equations with n unknowns,

$$\begin{aligned} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned} \tag{8}$$

the solution is written

$$x_i = \frac{\det A_i}{\det A} \;,\qquad i = 1, 2, \cdots, n \tag{9}$$

where A is the $n\times n$ matrix of the coefficients a_{jk} of the unknowns, while A_i is the matrix obtained from A by replacing the column of the coefficients of x_i with the column of the constants b_k .

We note the following:

1. If $\det A \neq 0$ (i.e., if the matrix A is invertible) a unique solution (9) of the system (8) exists.

2. If $\det A=0$ (the matrix A is *not* invertible) and if *even one* of the Cramer determinants $\det A_k$ in (9) is non-vanishing, the system (8) *has no solution* (is *inconsistent*), as follows from (9).

3. If $\det A=0$ and if *all* Cramer determinants $\det A_k$ ($k=1,2,...,n$) are zero, the system (8) has an *infinite number* of solutions.

Particularly significant for applications is the case of a *homogeneous* system, in which all constant terms b_k ($k=1,2,...,n$) are zero:

$$\begin{aligned}
&a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0\\
&a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0\\
&\vdots\\
&a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=0
\end{aligned} \tag{10}$$

In this case *all* Cramer determinants $\det A_k$ ($k=1,2,...,n$) are zero (explain this!). The following possibilities thus exist:

1. If the determinant of the matrix A of the coefficients of the unknowns is non-zero ($\det A\neq 0$), the only possible solution of the system (10) is the *trivial solution* $x_1 = x_2 = \ldots = x_n = 0$, as follows from (9).

2. If $\det A=0$, the system (10) admits an *infinite number* of nontrivial solutions.

Exercise: Show the following: (*a*) A homogeneous linear system always has a solution, i.e., is never inconsistent. (*b*) For such a system to possess a *nontrivial* solution (different, that is, from the zero solution) the determinant of the matrix of coefficients of the unknowns must be zero.

Example: Consider the homogeneous system

$$\begin{aligned}
2x - y &= 0\\
-6x + 3y &= 0
\end{aligned}$$

(where we have put $x_1=x$, $x_2=y$). The determinant of the coefficients of the unknowns is

$$\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 6-6=0 \ .$$

This occurs because the second line is a multiple (by –3) of the first. And this, in turn, reflects the fact that the equations in the system *are not independent* of each other (the second one is just a multiple of the first, thus does not provide any useful new information). The only thing we can say is that y=2x, with *arbitrary* x. This means that the system has an *infinite number* of solutions, one for each chosen value of x.

APPLICATION TO THE VECTOR PRODUCT

Consider the vectors

$$\vec{A} = A_x\,\hat{u}_x + A_y\,\hat{u}_y + A_z\,\hat{u}_z \equiv (A_x, A_y, A_z) \ ,$$
$$\vec{B} = B_x\,\hat{u}_x + B_y\,\hat{u}_y + B_z\,\hat{u}_z \equiv (B_x, B_y, B_z) \ ,$$

where $\hat{u}_x, \hat{u}_y, \hat{u}_z$ are the *unit vectors* on the axes *x, y, z*, respectively, of a standard Cartesian system. As we know from vector analysis, the *vector product* (or “cross product”) of $\vec{A}$ and $\vec{B}$ can be written in determinant form, as follows:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \ .$$

Moreover, the necessary condition in order that $\vec{A}$ and $\vec{B}$ be *parallel* to each other is $\vec{A}\times\vec{B}=0$.

Example: Find the values of α and β for which the vectors $\vec{A}\equiv(1,\alpha,3)$ and $\vec{B}\equiv(-2,-4,\ \beta)$ are parallel to each other.

Solution: We must have $\vec{A}\times\vec{B}=0 \ \Rightarrow$

$$\begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ 1 & \alpha & 3 \\ -2 & -4 & \beta \end{vmatrix} = 0 \ \Rightarrow \ \hat{u}_x\,(\alpha\beta+12) - \hat{u}_y\,(\beta+6) + \hat{u}_z\,(-4+2\alpha) = 0$$

(where the determinant has been developed with respect to the first row, i.e., the row of the unit vectors). Given that the unit vectors constitute a linearly independent set, the only way the above equality may be satisfied is by setting all three coefficients of the corresponding unit vectors equal to zero. We thus obtain a system of *three* equations with *two* unknowns:

$$2\alpha - 4 = 0 \,, \qquad \beta + 6 = 0 \,, \qquad \alpha\beta + 12 = 0 \,.$$

The first two equations yield $\alpha=2, \ \beta=-6$. The third equation simply verifies this result. That is, the third equation is *compatible* with the other two but furnishes no additional information, since this last equation *is not independent* of the preceding ones but follows directly from them. Note that, with the values of α and β found above, the third row of the determinant that represents $\vec{A}\times\vec{B}$ becomes a multiple (by –2) of the second row, so that the determinant automatically vanishes.

Exercise: Show that no values of α and β exist for which the vectors $\vec{A}\equiv(1,\alpha,3)$ and $\vec{B}\equiv(-2,\beta,6)$ are parallel to each other.

Exercise: Show that there is an *infinite* number of values of α and β for which the vectors $\vec{A}\equiv(1,\alpha,3)$ and $\vec{B}\equiv(-2,\beta,-6)$ are parallel to each other. What relation must exist between α and β?