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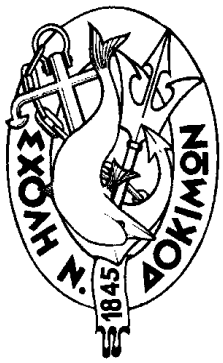

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Foundations of Newtonian Dynamics: An Axiomatic Approach for the Thinking Student

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Despite its apparent simplicity, Newtonian Mechanics contains conceptual subtleties that may cause some confusion to the deep-thinking student. These subtleties concern fundamental issues such as, e.g., the number of independent laws needed to formulate the theory, or, the distinction between genuine physical laws and derivative theorems. This article attempts to clarify these issues for the benefit of the student by revisiting the foundations of Newtonian Dynamics and by proposing a rigorous axiomatic approach to the subject. This theoretical scheme is built upon two fundamental postulates, namely, conservation of momentum and superposition property for interactions. Newton's Laws, as well as all familiar theorems of Mechanics, are shown to follow from these basic principles.

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Foundations of Newtonian Dynamics: An Axiomatic Approach for the Thinking Student [1]

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Abstract. Despite its apparent simplicity, Newtonian mechanics contains conceptual subtleties that may cause some confusion to the deep-thinking student. These subtleties concern fundamental issues such as, e.g., the number of independent laws needed to formulate the theory, or, the distinction between genuine physical laws and derivative theorems. This article attempts to clarify these issues for the benefit of the student by revisiting the foundations of Newtonian dynamics and by proposing a rigorous axiomatic approach to the subject. This theoretical scheme is built upon two fundamental postulates, namely, conservation of momentum and superposition property for interactions. Newton's laws, as well as all familiar theorems of mechanics, are shown to follow from these basic principles.

1. Introduction

Teaching introductory mechanics can be a major challenge, especially in a class of students that are not willing to take anything for granted! The problem is that, even some of the most prestigious textbooks on the subject may leave the student with some degree of confusion, which manifests itself in questions like the following:

- Is the law of inertia (Newton's first law) a law of motion (of free bodies) or is it a statement of existence (of inertial reference frames)?
- Are the first two of Newton's laws independent of each other? It appears that the first law is redundant, being no more than a special case of the second law!
- Is the second law a true law or a definition (of force)?
- Is the third law more fundamental than conservation of momentum, or is it the other way around?
- Does the "parallelogram rule" for composition of forces follow trivially from Newton's laws, or is an additional, independent principle required?
- And, finally, what is the minimum number of *independent* laws needed in order to build a complete theoretical basis for mechanics?

In this article we describe an axiomatic approach to introductory mechanics that is both rigorous and pedagogical. It purports to clarify issues like the ones mentioned above, at an early stage of the learning process, thus aiding the student to acquire a deep understanding of the basic ideas of the theory. It is not the purpose of this article, of course, to present an outline of a complete course of mechanics! Rather, we will focus on the most fundamental concepts and principles, those that are taught at the early chapters of dynamics (we will not be concerned with kinematics, since this subject confines itself to a description of motion rather than investigating the physical laws governing this motion).

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[1] See Note at the end of the article.

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The axiomatic basis of our approach consists of two fundamental postulates, presented in Section 3. The first postulate (*P1*) embodies both the existence of *inertial reference frames* and the *conservation of momentum*, while the second one (*P2*) expresses a *superposition principle for interactions*. The *law of inertia* is deduced from *P1*.

In Sec. 4, the concept of *force* on a particle subject to interactions is defined (as in *Newton's second law*) and *P2* is used to show that a composite interaction of a particle with others is represented by a vector sum of forces. Then, *P1* and *P2* are used to derive the *action-reaction law*. Finally, a generalization to systems of particles subject to external interactions is made.

For completeness of presentation, certain derivative concepts such as angular momentum, work, kinetic energy, etc., are discussed in Sec. 5. To make the article self-contained, proofs of all theorems are included.

2. A critical look at Newton's theory

There have been several attempts to reexamine Newton's laws even since Newton's time. Probably the most important revision of Newton's ideas – and the one on which modern mechanics teaching is based – is that due to Ernst Mach (1838-1916) (for a beautiful discussion of Mach's ideas, see the classic article by H. A. Simon [1]). Our approach differs in several aspects from those of Mach and Simon, although all these approaches share common characteristics in spirit. (For a historical overview of the various viewpoints regarding the theoretical basis of classical mechanics, see, e.g., the first chapter of [2].)

The question of the *independence* of Newton's laws has troubled many generations of physicists. In particular, still on this day some authors assert that the first law (the law of inertia) is but a special case of the second law. The argument goes as follows:

> *"According to the second law, the acceleration of a particle is proportional to the total force acting on it. Now, in the case of a free particle the total force on it is zero. Thus, a free particle must not be accelerating, i.e., its velocity must be constant. But, this is precisely what the law of inertia says!"*

Where is the error in this line of reasoning? Answer: The error rests in regarding the acceleration as an absolute quantity independent of the observer that measures it. As we well know, this is not the case. In particular, the only observer *entitled* to conclude that a non-accelerating object is subject to no net force is an *inertial observer*, one who uses an *inertial frame of reference* for his/her measurements. It is precisely the law of inertia that *defines* inertial frames and *guarantees* their existence. So, without the first law, the second law becomes indeterminate, if not altogether wrong, since it would appear to be valid relative to any observer regardless of his/her state of motion. It may be said that the first law defines the "terrain" within which the second law acquires a meaning. Applying the latter law without taking the former one into account would be like trying to play soccer without possessing a soccer field!

The completeness of Newton's laws is another issue. Let us see a significant example: As is well known, the *principle of conservation of momentum* is a direct consequence of Newton's laws. This principle dictates that the total momentum of a system of particles is constant in time, relative to an inertial frame of reference, when the

total external force on the system vanishes (in particular, this is true for an *isolated* system of particles, i.e., a system subject to no external forces). But, when proving this principle we take it for granted that the total force on each particle is the vector sum of all forces (both internal and external) acting on it. This is *not* something that follows trivially from Newton's laws, however! In fact, it was Daniel Bernoulli who first stated this *principle of superposition* after Newton's death. This means that classical Newtonian mechanics is built upon a total of *four* – rather than just three – basic laws.

The question now is: can we somehow "compactify" the axiomatic basis of Newtonian mechanics in order for it to consist of a smaller number of independent principles? At this point it is worth taking a closer look at the principle of conservation of momentum mentioned above. In particular, we note the following:

- For an isolated "system" consisting of a single particle, conservation of momentum reduces to the law of inertia (the momentum, thus also the velocity, of a free particle is constant relative to an inertial frame of reference).
- For an isolated system of two particles, conservation of momentum takes us back to the action-reaction law (Newton's third law).

Thus, starting with four fundamental laws (the three laws of Newton plus the law of superposition) we derived a new principle (conservation of momentum) that yields, as special cases, two of the laws we started with. The idea is then that, by taking *this* principle as our fundamental physical law, the number of independent laws necessary for building the theory would be reduced.

How about Newton's second law? We take the view, adopted by several authors including Mach himself (see, e.g., [1,3-7]) that this "law" should be interpreted as the *definition* of force in terms of the rate of change of momentum.

We thus end up with a theory built upon *two* fundamental principles, i.e., the conservation of momentum and the principle of superposition. In the following sections these ideas are presented in more detail.

3. The fundamental postulates and their consequences

We begin with some basic definitions.

Definition 1. A *frame of reference* (or *reference frame*) is a system of coordinates (or axes) used by an observer to measure physical quantities such as the position, the velocity, the acceleration, etc., of any particle in space. The position of the observer him/herself is assumed *fixed* relative to his/her own frame of reference.

Definition 2. An *isolated system of particles* is a system of particles subject only to their mutual interactions, i.e., subject to no *external* interactions. Any system of particles subject to external interactions that somehow cancel one another in order to make the system's motion identical to that of an isolated system will also be considered "isolated". In particular, an isolated system consisting of a single particle is called a *free particle*.

Our first fundamental postulate of mechanics is stated as follows:

Postulate 1. A class of frames of reference (*inertial frames*) exists such that, for any *isolated* system of particles, a vector equation of the following form is valid:

$$\sum_i m_i \vec{v}_i = \text{constant in time} \tag{1}$$

where $\vec{v}_i$ is the velocity of the particle indexed by i ($i = 1, 2, \cdots$) and where m_i is a constant quantity associated with this particle, which quantity is independent of the number or the nature of interactions the particle is subject to.

We call m_i the *mass* and $\vec{p}_i = m_i \vec{v}_i$ the *momentum* of the ith particle. Also, we call

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \tag{2}$$

the *total momentum* of the system relative to the considered reference frame. Postulate 1, then, expresses the *principle of conservation of momentum*: the total momentum of an isolated system of particles, relative to an inertial reference frame, is constant in time. (The same is true, in particular, for a free particle.)

Corollary 1. A free particle moves with constant velocity (i.e., with no acceleration) relative to an *inertial* reference frame.

Corollary 2. Any two free particles move with constant velocities relative to each other (their relative velocity is constant and their relative acceleration is zero).

Corollary 3. The position of a free particle may define the origin of an inertial frame of reference.

We note that Corollaries 1 and 2 constitute alternate expressions of the *law of inertia* (*Newton's first law*).

By *inertial observer* we mean an "intelligent" free particle, i.e., one that can perform measurements of physical quantities such as velocity or acceleration. By convention, the observer is assumed to be located at the origin of his/her own inertial frame of reference.

Corollary 4. Inertial observers move with constant velocities (i.e., they do not accelerate) relative to one another.

Consider now an isolated system of two particles of masses m_1 and m_2. Assume that the particles are allowed to interact for some time interval Δt. By conservation of momentum relative to an inertial frame of reference, we have:

$$\Delta(\vec{p}_1 + \vec{p}_2) = 0 \;\Rightarrow\; \Delta\vec{p}_1 = -\Delta\vec{p}_2 \;\Rightarrow\; m_1 \Delta\vec{v}_1 = -m_2 \Delta\vec{v}_2 \ .$$

We note that the changes in the velocities of the two particles within the (arbitrary) time interval Δt must be in opposite directions, a fact that is verified experimentally. Moreover, these changes are independent of the particular inertial frame used to measure the velocities (although, of course, the velocities themselves *are* frame-

dependent!). This latter statement is a consequence of the constancy of the relative velocity of any two inertial observers (the student is invited to explain this in detail). Now, taking magnitudes in the above vector equation, we have:

$$\frac{|\Delta\vec{v}_1|}{|\Delta\vec{v}_2|} = \frac{m_2}{m_1} = \text{constant} \tag{3}$$

regardless of the kind of interaction or the time Δt (which also is an experimentally verified fact). These demonstrate, in practice, the validity of the first postulate. Equation (3) allows us to specify the mass of a particle numerically, relative to the mass of some other particle (which particle may arbitrarily be assigned a unit mass), by letting the two particles interact for some time. As argued above, the result will be independent of the specific inertial frame used by the observer who makes the measurements. That is, in the classical theory, *mass is a frame-independent quantity*.

So far we have examined the case of isolated systems and, in particular, free particles. Consider now a particle subject to interactions with the rest of the world. Then, in general (unless these interactions somehow cancel one another), the particle's momentum will not remain constant relative to an *inertial* reference frame, i.e., will be a function of time. Our second postulate, which expresses the *superposition principle for interactions*, asserts that external interactions act on a particle *independently of one another* and their effects are superimposed.

Postulate 2. If a particle of mass m is subject to interactions with particles $m_1, m_2, \cdots$, then, at each instant t, the rate of change of this particle's momentum relative to an inertial reference frame is equal to

$$\frac{d\vec{p}}{dt} = \sum_i \left(\frac{d\vec{p}}{dt}\right)_i \tag{4}$$

where $\left(d\vec{p}/dt\right)_i$ is the rate of change of the particle's momentum due solely to the interaction of this particle with the particle m_i (i.e., the rate of change of $\vec{p}$ *if* the particle m interacted *only* with m_i).

4. The concept of force and the Third Law

We now *define* the concept of force, in a manner similar to *Newton's second law*:

Definition 3. Consider a particle of mass m that is subject to interactions. Let $\vec{p}(t)$ be the particle's momentum as a function of time, as measured relative to an *inertial* reference frame. The vector quantity

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{5}$$

is called the *total force* acting on the particle at time t.

Taking into account that, for a single particle, $\vec{p} = m\vec{v}$ with fixed *m*, we may rewrite Eq. (5) in the equivalent form,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \tag{6}$$

where $\vec{a}$ is the particle's acceleration at time *t*. Given that both the mass and the acceleration (prove this!) are independent of the inertial frame used to measure them, we conclude that *the total force on a particle is a frame-independent quantity*.

Corollary 5. Consider a particle of mass *m* subject to interactions with particles $m_1, m_2, \cdots$. Let $\vec{F}$ be the total force on *m* at time *t*, and let $\vec{F}_i$ be the force on *m* due solely to its interaction with m_i. Then, by the superposition principle for interactions (Postulate 2) as expressed by Eq. (4), we have:

$$\vec{F} = \sum_i \vec{F}_i \tag{7}$$

Theorem 1. Consider two particles *1* and *2*. Let $\vec{F}_{12}$ be the force on particle *1* due to its interaction with particle *2* at time *t*, and let $\vec{F}_{21}$ be the force on particle *2* due to its interaction with particle *1* at the same instant. Then,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{8}$$

Proof. By the independence of interactions, as expressed by the superposition principle, the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are independent of the presence or not of other particles in interaction with particles *1* and *2*. Thus, without loss of generality, we may assume that the system of the two particles is isolated. Then, by conservation of momentum and by using Eq. (5),

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0 \;\Rightarrow\; \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \;\Rightarrow\; \vec{F}_{12} = -\vec{F}_{21}\ .$$

Equation (8) expresses the *action-reaction law* (*Newton's third law*).

Theorem 2. The rate of change of the total momentum $\vec{P}(t)$ of a system of particles, relative to an inertial frame of reference, equals the total *external* force acting on the system at time *t*.

Proof. Consider a system of particles of masses m_i $(i = 1, 2, \cdots)$. Let $\vec{F}_i$ be the total *external* force on m_i (due to its interactions with particles *not belonging* to the system), and let $\vec{F}_{ij}$ be the *internal* force on m_i due to its interaction with m_j (by convention, $\vec{F}_{ij} = 0$ when *i=j*). Then, by Eq. (5) and by taking into account Eq. (7),

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i + \sum_j \vec{F}_{ij} \,.$$

By using Eq. (2) for the total momentum, we have:

$$\frac{d\vec{P}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i + \sum_{ij} \vec{F}_{ij} \,.$$

But,

$$\sum_{ij} \vec{F}_{ij} = \sum_{ji} \vec{F}_{ji} = \frac{1}{2} \sum_{ij} \left(\vec{F}_{ij} + \vec{F}_{ji} \right) = 0 \,,$$

where the action-reaction law (8) has been taken into account. So, finally,

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_i = \vec{F}_{ext} \tag{9}$$

where $\vec{F}_{ext}$ represents the *total external force* on the system.

5. Derivative concepts and theorems

Having presented the most fundamental concepts of mechanics, we now turn to some useful derivative concepts and related theorems, such as those of angular momentum and its relation to torque, work and its relation to kinetic energy, and conservative force fields and their association with mechanical-energy conservation.

Definition 4. Let *O* be the origin of an *inertial* reference frame, and let $\vec{r}$ be the position vector of a particle of mass *m*, relative to *O*. The vector quantity

$$\vec{L} = \vec{r} \times \vec{p} = m \left(\vec{r} \times \vec{v} \right) \tag{10}$$

(where $\vec{p} = m\vec{v}$ is the particle's momentum in the considered frame) is called the *angular momentum* of the particle relative to *O*.

Theorem 3. The rate of change of the angular momentum of a particle, relative to *O*, is given by

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \equiv \vec{T} \tag{11}$$

where $\vec{F}$ is the *total* force on the particle at time *t* and where $\vec{T}$ is the *torque* of this force relative to *O*, at this instant.

Proof. Equation (11) is easily proven by differentiating Eq. (10) with respect to time and by using Eq. (5).

Corollary 6. If the torque of the total force on a particle, relative to some point *O*, vanishes, then the angular momentum of the particle relative to *O* is constant in time (*principle of conservation of angular momentum*).

Under appropriate conditions, the above conservation principle can be extended to the more general case of a system of particles (see, e.g., [2-8]).

Definition 5. Consider a particle of mass *m* in a *force field* $\vec{F}(\vec{r})$, where $\vec{r}$ is the particle's position vector relative to the origin *O* of an inertial reference frame. Let *C* be a curve representing the trajectory of the particle from point *A* to point *B* in this field. Then, the line integral

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} \tag{12}$$

represents the *work* done by the force field on *m* along the path *C*. (Note: This definition is valid independently of whether or not additional forces, not related to the field, are acting on the particle; i.e., regardless of whether or not $\vec{F}(\vec{r})$ represents the total force on *m*.)

Theorem 4. Let $\vec{F}(\vec{r})$ represent the *total* force on a particle of mass *m* in a force field. Then, the work done on the particle along a path *C* from *A* to *B* is equal to

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = E_{k,B} - E_{k,A} = \Delta E_k \tag{13}$$

where

$$E_k = \frac{1}{2} m v^2 = \frac{p^2}{2m} \tag{14}$$

is the *kinetic energy* of the particle.

Proof. By using Eq. (6), we have:

$$\vec{F} \cdot d\vec{r} = m \frac{d\vec{v}}{dt} \cdot d\vec{r} = m\vec{v} \cdot d\vec{v} = \frac{1}{2} m\, d(\vec{v} \cdot \vec{v}) = \frac{1}{2} m\, d(v^2) = m v dv\,,$$

from which Eq. (13) follows immediately.

Definition 6. A force field $\vec{F}(\vec{r})$ is said to be *conservative* if a scalar function $E_p(\vec{r})$ (*potential energy*) exists, such that the work on a particle along *any* path from *A* to *B* can be written as

$$W_{AB} = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = E_{p,A} - E_{p,B} = -\Delta E_p \tag{15}$$

Theorem 5. If the total force $\vec{F}(\vec{r})$ acting on a particle *m* is conservative, with an associated potential energy $E_p(\vec{r})$, then the quantity

$$E = E_k + E_p = \frac{1}{2} m v^2 + E_p(\vec{r}) \tag{16}$$

(*total mechanical energy* of the particle) remains constant along any path traced by the particle (*conservation of mechanical energy*).

Proof. By combining Eq. (13) (which is generally valid for *any* kind of force) with Eq. (15) (which is valid for *conservative* force fields) we find:

$$\Delta E_k = -\Delta E_p \;\Rightarrow\; \Delta (E_k + E_p) = 0 \;\Rightarrow\; E_k + E_p = const.$$

Theorems 4 and 5 are readily extended to the case of a system of particles (see, e.g., [2-8]).

6. Some conceptual problems

After establishing our axiomatic basis and demonstrating that the standard Newtonian laws are consistent with it, the development of the rest of mechanics follows familiar paths. Thus, as we saw in the previous section, we can define concepts such as angular momentum, work, kinetic and total mechanical energies, etc., and we can state derivative theorems such as conservation of angular momentum, conservation of mechanical energy, etc. Also, rigid bodies and continuous media can be treated in the usual way [2-8] as systems containing an arbitrarily large number of particles.

Despite the more "economical" axiomatic basis of Newtonian mechanics suggested here, however, certain problems inherent in the classical theory remain. Let us point out a few:

1. The problem of "inertial frames"

An inertial frame of reference is only a theoretical abstraction: such a frame cannot exist in reality. As follows from the discussion in Sec. 3, the origin (say, *O*) of an inertial frame coincides with the position of a hypothetical free particle and, moreover, any real free particle moves with constant velocity relative to *O*. However, no such thing as an absolutely free particle may exist in the world. In the first place, every material particle is subject to the infinitely long-range gravitational interaction with the rest of the world. Furthermore, in order for a supposedly inertial observer to measure the velocity of a "free" particle and verify that this particle is not accelerating relative to him/her, the observer must somehow interact with the particle. Thus, no matter how weak this interaction may be, the particle cannot be considered free in the course of the observation.

2. The problem of simultaneity

In Sec. 4 we used our two postulates, together with the definition of force, to derive the action-reaction law. Implicit in our arguments was the requirement that action

must be *simultaneous* with reaction. As is well known, this hypothesis, which suggests instantaneous action at a distance, ignores the finite speed of propagation of the field associated with the interaction and violates causality.

3. A dimensionless "observer"

As we have used this concept, an "observer" is an intelligent free particle capable of making measurements of physical quantities such as velocity or acceleration. Such an observer may use any convenient (preferably rectangular) set of axes (x, y, z) for his/her measurements. Different systems of axes used by this observer have different orientations in space. By convention, the observer is located at the origin O of the chosen system of axes.

As we know, inertial observers do not accelerate relative to one another. Thus, the relative velocity of the origins (say, O and O') of two different inertial frames of reference is constant in time. But, what if the axes of these frames are in *relative rotation* (although the origins O and O' move uniformly relative to each other, or even coincide)? How can we tell which observer (if any) is an inertial one?

The answer is that, relative to the system of axes of an inertial frame, a free particle does not accelerate. In particular, relative to a rotating frame, a free particle will appear to possess at least a centripetal acceleration. Such a frame, therefore, cannot be inertial.

As mentioned previously, an object with finite dimensions (e.g., a rigid body) can be treated as an arbitrarily large system of particles. No additional postulates are thus needed in order to study the dynamics of such an object. This allows us to regard momentum and its conservation as more fundamental than angular momentum and its conservation, respectively. In this regard, our approach differs significantly from, e.g., that of Simon [1] who, in his own treatment, places the aforementioned two conservation laws on an equal footing from the outset.

7. Summary

Newtonian mechanics is the first subject in Physics an undergraduate student is exposed to. It continues to be important even at the intermediate and advanced levels, despite the predominant role played there by the more general formulations of Lagrangian and Hamiltonian dynamics.

It is this author's experience as a teacher that, despite its apparent simplicity, Newtonian mechanics contains certain conceptual subtleties that may leave the deep-thinking student with some degree of confusion. The average student, of course, is happy with the idea that the whole theory is built upon three rather simple laws attributed to Newton's genius. In the mind of the more demanding student, however, puzzling questions often arise, such as, e.g., how many independent laws we really need to fully formulate the theory, or, which ones should be regarded as truly fundamental laws of Nature, as opposed to others that can be derived as theorems.

This article suggested an axiomatic approach to introductory mechanics, based on two fundamental, empirically verifiable laws; namely, the *principle of conservation of momentum* and the *principle of superposition for interactions*. We showed that all standard ideas of mechanics (including, of course, Newton's laws) naturally follow from these basic principles. To make our formulation as economical as possible, we expressed the first principle in terms of a system of particles and treated the single-

particle situation as a special case. To make the article self-contained for the benefit of the student, explicit proofs of all theorems were given.

By no means do we assert, of course, that this particular approach is unique or pedagogically superior to other established methods that adopt different viewpoints regarding the axiomatic basis of classical mechanics. Moreover, as noted in Sec. 6, this approach is not devoid of the usual theoretical problems inherent in Newtonian mechanics (see also [9,10]).

In any case, it looks like classical mechanics remains a subject open to discussion and re-interpretation, and more can always be said about things that are usually taken for granted by most students (this is not exclusively their fault, of course!). Happily, some of my own students do not fall into this category. I appreciate the hard time they enjoy giving me in class!

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[3] Available on the Internet; please search title.

Electromotive Force: A Guide for the Perplexed

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Abstract. The concept of electromotive force (emf) may be introduced in various ways in an undergraduate course of theoretical electromagnetism. The multitude of alternate expressions for the emf is often the source of confusion to the student. We summarize the main ideas, adopting a pedagogical logic that proceeds from the general to the specific. The emf of a "circuit" is first defined in the most general terms. The expressions for the emf of some familiar electrodynamical systems are then derived in a rather straightforward manner. A diversity of physical situations is thus unified within a common theoretical framework.

1. INTRODUCTION

The difficulty in writing this article was not just due to the subject itself: we had to first overcome some almost irreconcilable differences in educational philosophy between an (opinionated) theoretical physicist and an (equally -if not more- opinionated) electrical engineer. At long last, a compromise was reached! This paper is the fruit of this "mutual understanding".

Having taught intermediate-level electrodynamics courses for several years, we have come to realize that, in the minds of many of our students, the concept of *electromotive force* (*emf*) is something of a mystery. What is an emf, after all? Is it the voltage of an ideal battery in a DC circuit? Is it work per unit charge? Or is it, in a more sophisticated way, the line integral of the electric field along a closed path? And what if a magnetic rather than an electric field is present?

Generally speaking, the problem with the emf lies in the diversity of situations where this concept applies, leading to a multitude of corresponding expressions for the emf. The subject is discussed in detail, of course, in all standard textbooks on electromagnetism, both at the intermediate [1-9] and at the advanced [10-12] level. Here we summarize the main ideas, choosing a pedagogical approach that proceeds from the general to the specific. We begin by defining the concept of emf of a "circuit" in the most general way possible. We then apply this definition to certain electrodynamic systems in order to recover familiar expressions for the emf. The main advantage of this approach is that a number of different physical situations are treated in a unified way within a common theoretical framework.

The general definition of the emf is given in Section 2. In subsequent sections (Sec.3-5) application is made to particular cases, such as motional emf, the emf due to a time-varying magnetic field, and the emf of a DC circuit consisting of an ideal battery and a resistor. In Sec.6, the connection between the emf and Ohm's law is discussed.

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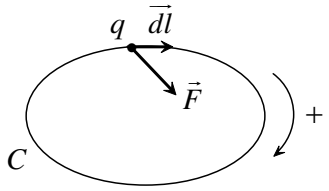
2. THE GENERAL DEFINITION OF EMF

Consider a region of space in which an electromagnetic (e/m) field exists. In the most general sense, any *closed* path *C* (or *loop*) within this region will be called a *"circuit"* (whether or not the whole or parts of *C* consist of material objects such as wires, resistors, capacitors, batteries, or any other elements whose presence may contribute to the e/m field).

We *arbitrarily* assign a positive direction of traversing the loop *C*, and we consider an element $\overrightarrow{dl}$ of *C* oriented in the positive direction. Imagine now a test charge *q* located at the position of $\overrightarrow{dl}$, and let $\vec{F}$ be the force on *q* at time *t*:



This force is exerted by the e/m field itself, as well as, possibly, by additional energy *sources* (e.g., batteries) that can interact electrically with *q*. The *force per unit charge* at the position of $\overrightarrow{dl}$ at time *t*, is

$$\vec{f} = \frac{\vec{F}}{q} \tag{1}$$

Note that $\vec{f}$ is independent of *q*, since the force by the e/m field and/or the sources on *q* is proportional to the charge. In particular, reversing the sign of *q* will have no effect on $\vec{f}$ (although it will change the direction of $\vec{F}$).

We now define the *electromotive force* (*emf*) of the circuit *C* at time *t* as the line integral of $\vec{f}$ along *C*, taken in the *positive* sense of *C*:

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} \tag{2}$$

Note that the sign of the emf is dependent upon our choice of the positive direction of circulation of *C*: by changing this convention, the sign of $\mathcal{E}$ is reversed.

We remark that, in the *non-relativistic* limit, the emf of a circuit *C* is the same for all inertial observers since *at this limit* the force $\vec{F}$ is invariant under a change of frame of reference.

In the following sections we apply the defining equation (2) to a number of specific electrodynamic situations that are certainly familiar to the student.

3. MOTIONAL EMF IN THE PRESENCE OF A STATIC MAGNETIC FIELD

Consider a circuit consisting of a closed wire *C*. The wire is moving inside a *static* magnetic field $\vec{B}(\vec{r})$. Let $\vec{\upsilon}$ be the velocity of the element $\overrightarrow{dl}$ of *C* relative to our inertial frame of reference. A charge *q* (say, a free electron) at the location of $\overrightarrow{dl}$ executes a composite motion, due to the motion of the loop *C* itself relative to our frame, as well as the motion of *q along C*. The total velocity of *q* relative to us is $\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}'$, where $\vec{\upsilon}'$ is the velocity of *q* in a direction parallel to $\overrightarrow{dl}$. The force from the magnetic field on *q* is

$$\vec{F} = q\,(\vec{\upsilon}_{tot} \times \vec{B}) = q\,(\vec{\upsilon} \times \vec{B}) + q\,(\vec{\upsilon}' \times \vec{B}) \;\Rightarrow$$

$$\vec{f} = \frac{\vec{F}}{q} = (\vec{\upsilon} \times \vec{B}) + (\vec{\upsilon}' \times \vec{B})$$

By (2), then, the emf of the circuit *C* is

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl}$$

But, since $\vec{\upsilon}'$ is parallel to $\overrightarrow{dl}$, we have that $(\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl} = 0$. Thus, finally,

$$\mathcal{E} = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} \qquad (3)$$

Note that the wire *need not maintain a fixed shape, size or orientation* during its motion! Note also that the velocity $\vec{\upsilon}$ may vary around the circuit.

By using (3), it can be proven (see Appendix) that

$$\mathcal{E} = -\frac{d\Phi}{dt} \qquad (4)$$

where $\Phi = \int \vec{B} \cdot \overrightarrow{da}$ is the magnetic flux through the wire *C* at time *t*. Note carefully that (4) does not express any novel physical law: it is simply a direct consequence of the definition of the emf !

4. EMF DUE TO A TIME-VARYING MAGNETIC FIELD

Consider now a closed wire *C* that is *at rest* inside a *time-varying* magnetic field $\vec{B}(\vec{r},t)$. As experiments show, as soon as $\vec{B}$ starts changing, a current begins to flow in the wire. This looks impressive, given that the free charges in the (stationary) wire were initially at rest. And, as everybody knows, a magnetic field exerts forces on *moving* charges only! It is also observed experimentally that, if the magnetic field $\vec{B}$ stops varying in time, the current in the wire

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disappears. The only field that can put an initially stationary charge in motion and keep this charge moving is an *electric* field.

We are thus compelled to conclude that *a time-varying magnetic field is necessarily accompanied by an electric field.* (It is often said that "a changing magnetic field *induces* an electric field". This is somewhat misleading since it gives the impression that the "source" of an electric field could be a magnetic field. Let us keep in mind, however, that the true sources of any e/m field are the electric charges and the electric currents!)

So, let $\vec{E}(\vec{r},t)$ be the electric field accompanying the time-varying magnetic field $\vec{B}$. Consider again a charge *q* at the position of the element $\overrightarrow{dl}$ of the wire. Given that the wire is now at rest (relative to our inertial frame), the velocity of *q* will be due to the motion of the charge along the wire only, i.e., in a direction parallel to $\overrightarrow{dl}$: $\vec{\upsilon}_{tot} = \vec{\upsilon}'$ (since $\vec{\upsilon} = 0$). The force on *q* by the e/m field is

$$\vec{F} = q\,[\vec{E} + (\vec{\upsilon}_{tot} \times \vec{B})] = q\,[\vec{E} + (\vec{\upsilon}' \times \vec{B})] \;\Rightarrow$$

$$\vec{f} = \frac{\vec{F}}{q} = \vec{E} + (\vec{\upsilon}' \times \vec{B})$$

The emf of the circuit *C* is now

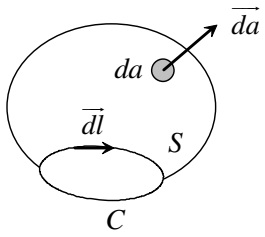
$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C \vec{E} \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl}$$

But, as explained earlier, $(\vec{\upsilon}' \times \vec{B}) \cdot \overrightarrow{dl} = 0$. Thus, finally,

$$\mathcal{E} = \oint_C \vec{E} \cdot \overrightarrow{dl} \qquad (5)$$

Equation (4) is still valid. This time, however, it is not merely a mathematical consequence of the definition of the emf ; rather, it is a true physical law deduced from experiment! Let us examine it in some detail.

In a region of space where a time-varying e/m field $(\vec{E},\vec{B})$ exists, consider an arbitrary open surface *S* bounded by the closed curve *C*:



(The *relative* direction of $\overrightarrow{dl}$ and the surface element $\overrightarrow{da}$, normal to *S*, is determined according to the familiar right-hand rule.) The loop *C* is assumed stationary relative to the inertial observer; hence the emf along *C* at time *t* is given by (5). The magnetic flux through *S* at this instant is

$$\Phi_m(t) = \int_S \vec{B} \cdot \overrightarrow{da}$$

(Note that the signs of $\mathcal{E}$ and Φ_m depend on the chosen positive direction of *C*.) Since the field $\vec{B}$ is *solenoidal*, the value of Φ_m for a given *C* is independent of the choice of the surface *S*. That is, the same magnetic flux will go through *any* open surface bounded by the closed curve *C*.

According to the *Faraday-Henry law*,

$$\mathcal{E} = -\frac{d\Phi_m}{dt} \tag{6}$$

or explicitly,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = -\frac{d}{dt}\int_S \vec{B} \cdot \overrightarrow{da} \tag{7}$$

(The negative sign on the right-hand sides of (6) and (7) expresses *Lenz's law*.)

Equation (7) can be re-expressed in differential form by using Stokes' theorem,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{da}$$

and by taking into account that the surface *S* may be arbitrarily chosen. The result is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8}$$

We note that if $\partial\vec{B}/\partial t \neq 0$, then necessarily $\vec{E} \neq 0$. Hence, as already mentioned, a time-varying magnetic field is always accompanied by an electric field. If, however, $\vec{B}$ is *static* ($\partial\vec{B}/\partial t = 0$), then $\vec{E}$ is *irrotational*: $\vec{\nabla} \times \vec{E} = 0 \Leftrightarrow \oint \vec{E} \cdot \overrightarrow{dl} = 0$, which allows for the possibility that $\vec{E} = 0$.

Corollary: The emf around a *fixed* loop *C* inside a *static* e/m field $\left(\vec{E}(\vec{r}), \vec{B}(\vec{r})\right)$ is $\mathcal{E} = 0$ (the student should explain this).

5. EMF OF A CIRCUIT CONTAINING A BATTERY AND A RESISTOR

Consider a circuit consisting of an ideal battery (i.e., one with no internal resistance) connected to an external resistor. As shown below, the emf of the circuit *in the direction of the current* is equal to the voltage *V* of the battery. Moreover, the emf in this case represents the *work per unit charge* done by the source (battery).

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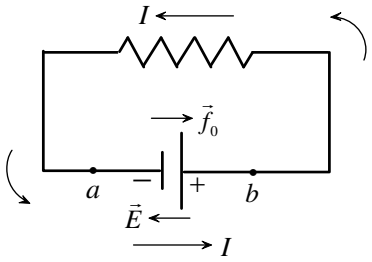

We recall that, in general, the emf of a circuit *C* at time *t* is equal to the integral

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl}$$

where $\vec{f} = \vec{F}/q$ is the force per unit charge at the location of the element $\overrightarrow{dl}$ of the circuit, at time *t*. In essence, we assume that in every element $\overrightarrow{dl}$ we have placed a test charge *q* (this could be, e.g., a free electron of the conducting part of the circuit). The force $\vec{F}$ on each *q* is then measured *simultaneously* for all charges at time *t*. Since here we are dealing with a *static* (time-independent) situation, however, we can treat the problem somewhat differently: The measurements of the forces $\vec{F}$ on the charges *q* need not be made at the same instant, given that nothing changes with time, anyway. So, instead of placing several charges *q* around the circuit and measuring the forces $\vec{F}$ on each of them at a particular instant, we imagine *a single charge q* making a complete tour around the loop *C*. We may assume, e.g., that the charge *q* is one of the (*conventionally positive*) free electrons taking part in the constant current *I* flowing in the circuit. We then measure the force $\vec{F}$ on *q* at each point of *C*.

We thus assume that *q* is a *positive* charge moving *in the direction of the current I*. We also assume that the direction of circulation of *C* is the *same as the direction of the current* (counterclockwise in the figure). During its motion, *q* is subject to two forces: (1) the force $\vec{F}_0$ by the source (battery) that carries *q* from the negative pole *a* to the positive pole *b through the source*, and (2) the electrostatic force $\vec{F}_e = q\vec{E}$ due to the electrostatic field $\vec{E}$ at each point of the circuit *C* (both inside and outside the source). The total force on *q* is

$$\vec{F} = \vec{F}_0 + \vec{F}_e = \vec{F}_0 + q\vec{E} \;\Rightarrow\; \vec{f} = \frac{\vec{F}}{q} = \frac{\vec{F}_0}{q} + \vec{E} \equiv \vec{f}_0 + \vec{E}$$

Then,

$$\mathcal{E} = \oint_C \vec{f} \cdot \overrightarrow{dl} = \oint_C \vec{f}_0 \cdot \overrightarrow{dl} + \oint_C \vec{E} \cdot \overrightarrow{dl} = \oint_C \vec{f}_0 \cdot \overrightarrow{dl} \qquad (9)$$

since $\oint_C \vec{E} \cdot \overrightarrow{dl} = 0$ for an electrostatic field. However, the action of the source on *q* is limited to the region between the poles of the battery, that is, the section of the circuit from *a* to *b*. Hence, $\vec{f}_0 = 0$ *outside* the source, so that (9) reduces to

$$\mathcal{E} = \int_a^b \vec{f}_0 \cdot \overrightarrow{dl} \qquad (10)$$

Now, since the current *I* is constant, the charge *q* moves at constant speed along the circuit. This means that the *total* force on *q* in the direction of the path *C* is zero. In the interior of the resistor, the electrostatic force $\vec{F}_e = q\vec{E}$ is counterbalanced by the force on *q* due to the collisions of the charge with the positive ions of the metal (this latter force does *not* contribute to the emf and is *not* counted in its evaluation!). In the interior of the (ideal) battery, however, where there is no resistance, the electrostatic force $\vec{F}_e$ must be counterbalanced by the *opposing* force $\vec{F}_0$ exerted by the source. Thus, in the section of the circuit between *a* and *b*,

$$\vec{F} = \vec{F}_0 + \vec{F}_e = 0 \;\Rightarrow\; \vec{f} = \frac{\vec{F}}{q} = \vec{f}_0 + \vec{E} = 0 \;\Rightarrow\; \vec{f}_0 = -\vec{E}$$

Equation (10) then takes the final form,

$$\mathcal{E} = -\int_a^b \vec{E}\cdot\overrightarrow{dl} = V_b - V_a = V \tag{11}$$

where V_a and V_b are the electrostatic potentials at *a* and *b*, respectively. This is, of course, what every student knows from elementary e/m courses!

The work done by the source on *q* upon transferring the charge from *a* to *b* is

$$W = \int_a^b \vec{F}_0\cdot\overrightarrow{dl} = q\int_a^b \vec{f}_0\cdot\overrightarrow{dl} = q\,\mathcal{E} \tag{12}$$

[where we have used (10)]. So, the *work of the source per unit charge* is $W/q = \mathcal{E}$. This work is converted into heat in the resistor, so that the source must again supply energy in order to carry the charges once more from *a* to *b*. This is something like the torture of Sisyphus in Greek mythology!

6. EMF AND OHM'S LAW

Consider a closed wire *C* inside an e/m field. The circuit may contain sources (e.g., a battery) and may also be in motion relative to our inertial frame of reference. Let *q* be a test charge at the location of the element $\overrightarrow{dl}$ of *C*, and let $\vec{F}$ be the total force on *q* (due to the e/m field and/or the sources) at time *t*. (As mentioned in Sec.2, this force is, classically, a frame-independent quantity.) The force per unit charge at the location of $\overrightarrow{dl}$ at time *t*, then, is $\vec{f} = \vec{F}/q$. According to our general definition, the emf of the circuit at time *t* is

$$\mathcal{E} = \oint_C \vec{f}\cdot\overrightarrow{dl} \tag{13}$$

Now, if σ is the *conductivity* of the wire, then, by *Ohm's law* in its general form (see, e.g., p. 285 of [1]) we have:

$$\vec{J} = \sigma\vec{f} \tag{14}$$

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where $\vec{J}$ is the *volume current density* at the location of $\overrightarrow{dl}$ at time *t*. (Note that the more common expression $\vec{J}=\sigma\vec{E}$, found in most textbooks, is a special case of the above formula. Note also that $\vec{J}$ is measured *relative to the wire*, thus is the same for all inertial observers.) By combining (13) and (14) we get:

$$\mathcal{E}=\frac{1}{\sigma}\oint_C \vec{J}\cdot\overrightarrow{dl} \tag{15}$$

Taking into account that $\vec{J}$ is in the direction of $\overrightarrow{dl}$ at each point of *C*, we write:

$$\vec{J}\cdot\overrightarrow{dl}=J\,dl=\frac{I}{S}dl$$

where *S* is the *constant* cross-sectional area of the wire. If we make the additional assumption that, at each instant *t*, the current *I* is constant around the circuit (although *I* may vary with time), we finally get:

$$\mathcal{E}=\frac{l}{\sigma S}I=\frac{\rho l}{S}I=IR \tag{16}$$

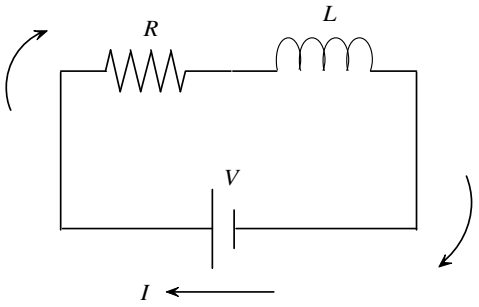
where *l* is the total length of the wire, $\rho=1/\sigma$ is the *resistivity* of the material, and *R* is the total resistance of the circuit. Equation (16) is the familiar special form of Ohm's law.

As an example, let us return to the circuit of Sec.5, this time assuming a *non-ideal* battery with internal resistance *r*. Let R_0 be the external resistance connected to the battery. The total resistance of the circuit is $R=R_0+r$. As before, we call $V=V_b-V_a$ the potential difference between the terminals of the battery, which is equal to the voltage across the external resistor. Hence, $V=IR_0$, where *I* is the current in the circuit. The emf of the circuit (in the direction of the current) is

$$\mathcal{E}=IR=I(R_0+r)=V+Ir$$

Note that the potential difference *V* between the terminals *a* and *b* equals the emf only when no current is flowing ($I=0$).

As another example, consider a circuit *C* containing an ideal battery of voltage *V* and having total resistance *R* and total *inductance L*:



In this case, the emf of *C in the direction of the current flow* is

$$\mathcal{E}(t) = V + V_L = V - L\frac{dI}{dt} = I(t)R$$

To understand why the total emf of the circuit is $V+V_L$, we think as follows: On its tour around the circuit, a test charge q is subject to two forces (ignoring collisions with the positive ions in the interior of the wire): a force *inside* the source, and a force by the *non-conservative* electric field accompanying the time-varying magnetic flux through the circuit. Hence, the total emf will be the sum of the emf due to the (ideal) battery alone and the emf expressed by the Faraday-Henry law (6). The latter emf is precisely V_L ; it has a nonzero value for as long as the current I is changing.

Some interesting energy considerations are here in order. The total power supplied to the circuit by the battery at time t is

$$P = IV = I^2R + LI\frac{dI}{dt}$$

The term I^2R represents the power *irreversibly lost* as heat in the resistor (energy, per unit time, spent in moving the electrons through the crystal lattice of the conductor and transferred to the ions that make up the lattice). Thus, this power must necessarily be supplied back by the source in order to *maintain* the current against dissipative losses in the resistor. On the other hand, the term $LI\,(dI/dt)$ represents the energy per unit time required to *build up* the current against the "back emf " V_L. This energy is *retrievable* and is given back to the source when the current decreases. It may also be interpreted as energy per unit time required in order to *establish the magnetic field* associated with the current. This energy is "stored" in the magnetic field surrounding the circuit.

7. CONCLUDING REMARKS

In concluding this article, let us highlight a few points of importance:

1. The emf was defined as a line integral of force per unit charge around a loop (or "circuit") in an e/m field. The loop may or may not consist of a real conducting wire, and it may contain sources such as batteries.

2. In the classical (non-relativistic) limit, the emf is independent of the inertial frame of reference with respect to which it is measured.

3. In the case of *purely motional* emf, Faraday's "law" (4) is in essence a mere consequence of the definition of the emf. On the contrary, when a time-dependent magnetic field is present, the similar-looking equation (6) is a true physical law (the Faraday-Henry law).

4. In a DC circuit with a battery, the emf in the direction of the current equals the voltage of the battery and represents work per unit charge done by the source.

5. If the loop describing the circuit represents a conducting wire of finite resistance, Ohm's law can be expressed in terms of the emf by equation (16).

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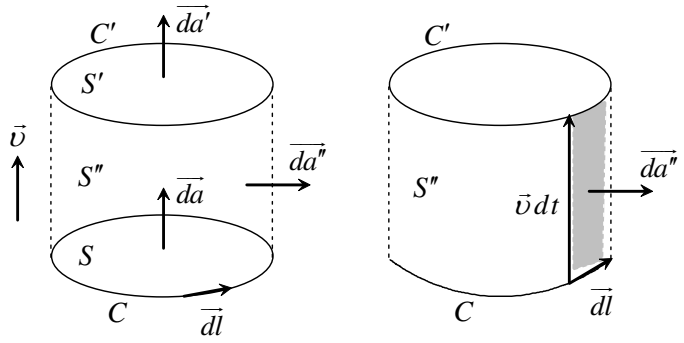
APPENDIX

Here is an analytical proof of equation (4) of Sec.3:

Assume that, at time *t*, the wire describes a closed curve *C* that is the boundary of a plane surface *S*. At time *t´= t+dt*, the wire (which has moved in the meanwhile) describes another curve *C´* that encloses a surface *S´*. Let $\overrightarrow{dl}$ be an element of *C* in the direction of circulation of the curve, and let $\vec{\upsilon}$ be the velocity of this element relative to an inertial observer (the velocity of the elements of *C* may vary along the curve):



The direction of the surface elements $\overrightarrow{da}$ and $\overrightarrow{da'}$ is consistent with the chosen direction of $\overrightarrow{dl}$, according to the right-hand rule. The element of the side ("cylindrical") surface *S´´* formed by the motion of *C*, is equal to

$$\overrightarrow{da''} = \overrightarrow{dl} \times (\vec{\upsilon}\, dt) = (\overrightarrow{dl} \times \vec{\upsilon})\, dt$$

Since the magnetic field is static, we can view the situation in a somewhat different way: Rather than assuming that the curve *C* moves within the time interval *dt* so that its points coincide with the points of the curve *C´* at time *t´*, we consider two *constant* curves *C* and *C´ at the same instant t*. In the case of a *static* field $\vec{B}$, the magnetic flux through *C´* at time *t´= t+dt* (according to our original assumption of a moving curve) is the same as the flux through this same curve at time *t*, given that no change of the magnetic field occurs within the time interval *dt*. Now, we note that the open surfaces S_1=*S* and S_2= *S´* ∪ *S´´* share a common boundary, namely, the curve *C*. Since the magnetic field is *solenoidal*, the same magnetic flux Φ_m passes through S_1 and S_2 at time *t*. That is,

$$\int_{S_1} \vec{B} \cdot \overrightarrow{da_1} = \int_{S_2} \vec{B} \cdot \overrightarrow{da_2} \;\Rightarrow\; \int_{S} \vec{B} \cdot \overrightarrow{da} = \int_{S'} \vec{B} \cdot \overrightarrow{da'} + \int_{S''} \vec{B} \cdot \overrightarrow{da''}$$

But, returning to our initial assumption of a *moving* curve, we note that

$$\int_{S} \vec{B} \cdot \overrightarrow{da} = \Phi_m(t) = \textit{magnetic flux through the wire at time } t$$

and

$\int_{S'} \vec{B} \cdot \overrightarrow{da'} = \Phi_m(t+dt) =$ *magnetic flux through the wire at time t+dt*

Hence,

$$\Phi_m(t) = \Phi_m(t+dt) + \int_{S''} \vec{B} \cdot \overrightarrow{da''} \Rightarrow$$

$$d\Phi_m = \Phi_m(t+dt) - \Phi_m(t) = -\int_{S''} \vec{B} \cdot \overrightarrow{da''} = -dt \oint_C \vec{B} \cdot (\overrightarrow{dl} \times \vec{\upsilon}) \Rightarrow$$

$$-\frac{d\Phi_m}{dt} = \oint_C \vec{B} \cdot (\overrightarrow{dl} \times \vec{\upsilon}) = \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} = \mathcal{E}$$

in accordance with (3) and (4).

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[1] One of us (C.J.P.) strongly feels that the 2nd Edition of 1975 (unfortunately out of print) was a much better edition!

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Does the electromotive force (always) represent work?

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Abstract

In the literature of Electromagnetism, the electromotive force of a "circuit" is often defined as work done on a unit charge during a complete tour of the latter around the circuit. We explain why this statement cannot be generally regarded as true, although it is indeed true in certain simple cases. Several examples are used to illustrate these points.

1. Introduction

In a recent paper [1] the authors suggested a pedagogical approach to the *electromotive force* (emf) of a "circuit", a fundamental concept of Electromagnetism. Rather than defining the emf in an *ad hoc* manner for each particular electrodynamic system, this approach begins with the most general definition of the emf and then specializes to certain cases of physical interest, thus recovering the familiar expressions for the emf.

Among the various examples treated in [1], the case of a simple battery-resistor circuit was of particular interest since, in this case, the emf was shown to be equal to the *work, per unit charge*, done by the source (battery) for a complete tour around the circuit. Now, in the literature of Electrodynamics the emf is often *defined* as work per unit charge. As we explain in this paper, this is not generally true except for special cases, such as the aforementioned one.

In Section 2, we give the general definition of the emf, $\mathcal{E}$, and, separately, that of the work per unit charge, w, done by the agencies responsible for the generation and preservation of a current flow in the circuit. We then state the necessary conditions in order for the equality $\mathcal{E}=w$ to hold. We stress that, by their very definitions, $\mathcal{E}$ and w are *different* concepts. Thus, the equation $\mathcal{E}=w$ suggests the possible equality of the *values* of two physical quantities, not the conceptual identification of these quantities!

Section 3 reviews the case of a circuit consisting of a battery connected to a resistive wire, in which case the equality $\mathcal{E}=w$ is indeed valid.

In Sec. 4, we study the problem of a wire moving through a static magnetic field. A particular situation where the equality $\mathcal{E}=w$ is valid is treated in Sec. 5.

Finally, Sec. 6 examines the case of a stationary wire inside a time-varying magnetic field. It is shown that the equality $\mathcal{E}=w$ is satisfied only in the special case where the magnetic field varies linearly with time.

2. The general definitions of emf and work per unit charge

Consider a region of space in which an electromagnetic (e/m) field exists. In the most general sense, any *closed* path C (or *loop*) within this region will be called a *"circuit"* (whether or not the whole or parts of C consist of material objects such as wires, resistors, capacitors, batteries, etc.). We *arbitrarily* assign a positive direction of traversing the loop C, and we consider an element $\vec{dl}$ of C oriented in the positive direction (Fig. 1).

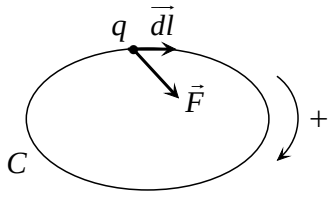

Figure 1: An oriented loop representing a circuit.

Imagine now a test charge q located at the position of $\vec{dl}$, and let $\vec{F}$ be the force on q at time t. This force is exerted by the e/m field itself, as well as, possibly, by additional *energy sources* (e.g., batteries or some external mechanical action) that may contribute to the generation and preservation of a current flow around the loop C. The *force per unit charge* at the position of $\vec{dl}$ at time t, is

$$\vec{f} = \frac{\vec{F}}{q} \tag{1}$$

Note that $\vec{f}$ is independent of q, since the electromagnetic force on q is proportional to the charge. In particular, reversing the sign of q will have no effect on $\vec{f}$ (although it will change the direction of $\vec{F}$).

In general, neither the shape nor the size of C is required to remain fixed. Moreover, the loop may be in motion rela-

tive to an external inertial observer. Thus, for a loop of (possibly) variable shape, size or position in space, we will use the notation $C(t)$ to indicate the state of the curve at time t.

We now define the *electromotive force* (emf) of the circuit C *at time* t as the line integral of $\vec{f}$ along C, taken in the *positive* sense of C:

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}(\vec{r},t)\cdot\vec{dl} \qquad (2)$$

(where $\vec{r}$ is the position vector of $\vec{dl}$ relative to the origin of our coordinate system). Note that the sign of the emf is dependent upon our choice of the positive direction of circulation of C: by changing this convention, the sign of $\mathcal{E}$ is reversed.

As mentioned above, the force (per unit charge) defined in (1) can be attributed to two factors: the interaction of q with the e/m field itself and the action on q due to any additional energy sources. Eventually, this latter interaction is *electromagnetic* in nature even when it originates from some external mechanical action. We write:

$$\vec{f} = \vec{f}_{em} + \vec{f}_{app} \qquad (3)$$

where $\vec{f}_{em}$ is the force due to the e/m field and $\vec{f}_{app}$ is the *applied force* due to an additional energy source. We note that the force (3) does not include any *resistive* (dissipative) forces that oppose a charge flow along C; it only contains forces that may contribute to the generation and preservation of such a flow in the circuit.

Now, suppose we allow *a single charge* q to make a full trip around the circuit C under the action of the force (3). In doing so, the charge describes a curve C' *in space* (not necessarily a closed one!) relative to an external inertial observer. Let $\vec{dl'}$ be an element of C' representing an infinitesimal displacement of q in space, in time dt. We define the *work per unit charge* for this complete tour around the circuit by the integral:

$$w = \int_{C'} \vec{f}\cdot\vec{dl'} \qquad (4)$$

For a *stationary* circuit of *fixed* shape, C' coincides with the closed curve C and (4) reduces to

$$w = \oint_{C} \vec{f}\cdot\vec{dl} \qquad (\textit{fixed } C) \qquad (5)$$

It should be noted carefully that the integral (2) is evaluated *at a fixed time* t, while in the integrals (4) and (5) time is allowed to flow! In general, the value of w depends on the time t_0 and the point P_0 at which q starts its round trip on C. Thus, there is a certain ambiguity in the definition of work per unit charge. On the other hand, the ambiguity (so to speak) with respect to the emf is related to the dependence of the latter on time t.

The question now is: can the emf be equal *in value* to the work per unit charge, despite the fact that these quantities are defined differently? For the equality $\mathcal{E}=w$ to hold, both $\mathcal{E}$ and w must be defined unambiguously. Thus, $\mathcal{E}$ must be *constant*, independent of time ($d\mathcal{E}/dt=0$) while w must not depend on the initial time t_0 or the initial point P_0 of the round trip of q on C. These requirements are *necessary conditions* in order for the equality $\mathcal{E}=w$ to be meaningful.

In the following sections we illustrate these ideas by means of several examples. As will be seen, the satisfaction of the above-mentioned conditions is the exception rather than the rule!

3. A resistive wire connected to a battery

Consider a circuit consisting of an ideal battery (i.e., one with no internal resistance) connected to a metal wire of total resistance R (Fig. 2). As shown in [1] (see also [2]), the emf of the circuit *in the direction of the current* is equal to the voltage V of the battery. Moreover, the emf in this case represents the work, per unit charge, done by the source (battery). Let us review the proof of these statements.

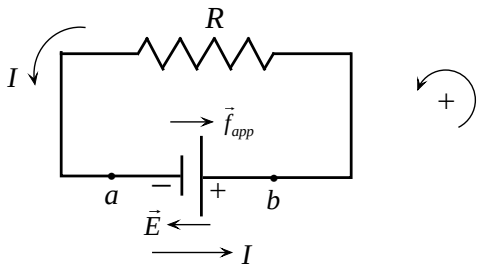

Figure 2: A battery connected to a resistive wire.

A (*conventionally positive*) moving charge q is subject to two forces around the circuit C: an electrostatic force $\vec{F}_e = q\vec{E}$ at every point of C and a force $\vec{F}_{app}$ *inside* the battery, the latter force carrying q from the negative pole a to the positive pole b *through the source*. According to (3), the total force per unit charge is

$$\vec{f} = \vec{f}_e + \vec{f}_{app} = \vec{E} + \vec{f}_{app} \ .$$

The emf in the direction of the current (i.e., counterclockwise), at any time t, is

$$\begin{aligned}\mathcal{E} &= \oint_C \vec{f}\cdot\vec{dl} \\ &= \oint_C \vec{E}\cdot\vec{dl} + \oint_C \vec{f}_{app}\cdot\vec{dl} \\ &= \int_a^b \vec{f}_{app}\cdot\vec{dl}\end{aligned} \qquad (6)$$

where we have used the facts that $\oint_C \vec{E}\cdot\vec{dl} = 0$ for an electrostatic field and that the action of the source on *q* is limited to the region between the poles of the battery.

Now, in a steady-state situation (*I* = constant) the charge *q* moves at constant speed along the circuit. This means that the total force on *q* in the direction of the path *C* is zero. In the interior of the wire, the electrostatic force $\vec{F}_e = q\vec{E}$ is counterbalanced by the resistive force on *q* due to the collisions of the charge with the positive ions of the metal (as mentioned previously, this latter force does *not* contribute to the emf). In the interior of the (ideal) battery, however, where there is no resistance, the electrostatic force must be counterbalanced by the opposing force exerted by the source. Thus, in the section of the circuit between *a* and *b*, $\vec{f}_{app} = -\vec{f}_e = -\vec{E}$. By (6), then, we have:

$$\mathcal{E} = -\int_a^b \vec{E}\cdot\vec{dl} = V_b - V_a = V \qquad (7)$$

where V_a and V_b are the electrostatic potentials at *a* and *b*, respectively. We note that the emf is constant in time, as expected in a steady-state situation.

Next, we want to find the work per unit charge for a complete tour around the circuit. To this end, we allow *a single charge q* to make a full trip around *C* and we use expression (5) (since the wire is stationary and of fixed shape). In applying this relation, time is assumed to flow as *q* moves along *C*. Given that the situation is static (time-independent), however, time is not really an issue since it doesn't matter at what moment the charge will pass by any given point of *C*. Thus, the integration in (5) will yield the same result (7) as the integration in (6), despite the fact that, in the latter case, time was assumed *fixed*. We conclude that the equality $w=\mathcal{E}$ is valid in this case: the emf *does* represent work per unit charge.

4. Moving wire inside a static magnetic field

Consider a wire *C* moving in the *xy*-plane. The shape and/or size of the wire need not remain fixed during its motion. A static magnetic field $\vec{B}(\vec{r})$ is present in the region of space where the wire is moving. For simplicity, we assume that this field is normal to the plane of the wire and directed *into* the page.

In Fig. 3, the *z*-axis is normal to the plane of the wire and directed towards the reader. We call $\vec{da}$ an infinitesimal normal vector representing an element of the plane surface bounded by the wire (this vector is directed *into* the plane, consistently with the chosen clockwise direction of traversing the loop *C*). If $\hat{u}_z$ is the unit vector on the *z*-axis, then $\vec{da} = -(da)\hat{u}_z$ and $\vec{B} = -B(\vec{r})\hat{u}_z$, where $B(\vec{r}) = |\vec{B}(\vec{r})|$.

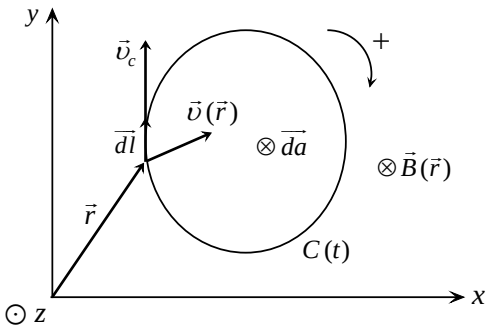

Figure 3: A wire *C* moving inside a static magnetic field.

Consider an element $\vec{dl}$ of the wire, located at a point with position vector $\vec{r}$ relative to the origin of our inertial frame of reference. Call $\vec{\upsilon}(\vec{r})$ the velocity of this element relative to our frame. Let *q* be a (*conventionally positive*) charge passing by the considered point at time *t*. This charge executes a composite motion, having a velocity $\vec{\upsilon}_c$ *along the wire* and acquiring an extra velocity $\vec{\upsilon}(\vec{r})$ due to the motion of the wire itself. The total velocity of *q* relative to us is $\vec{\upsilon}_{tot} = \vec{\upsilon}_c + \vec{\upsilon}$.

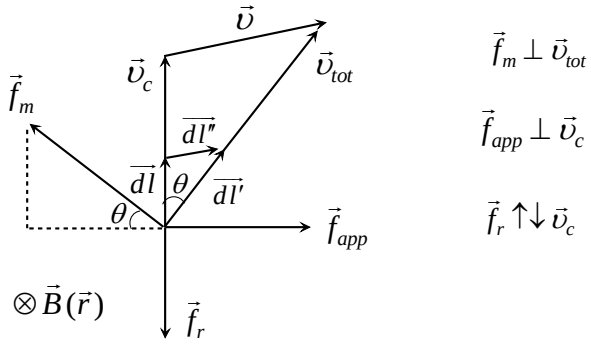

Figure 4: Balance of forces per unit charge.

The balance of forces acting on *q* is shown in the diagram of Fig. 4. The *magnetic force* on *q* is normal to the charge's total velocity and equal to $\vec{F}_m = q(\vec{\upsilon}_{tot}\times\vec{B})$. Hence, the magnetic force per unit charge is $\vec{f}_m = \vec{\upsilon}_{tot}\times\vec{B}$. Its component along the wire (i.e., in the direction of $\vec{dl}$) is counterbalanced by the *resistive force* $\vec{f}_r$, which opposes the motion of *q* along *C* (this force, as mentioned previously, does *not* contribute to the emf). However, the component of the magnetic force *normal* to the wire will tend to make the wire move "backwards" (in a direction opposing the desired motion of the wire) unless it is counterbalanced by some *external* mechanical action (e.g., our hand, which pulls the wire forward). Now, the charge *q* takes a share of this action by means of some force transferred to it by the structure of the wire. This force (which will be called an *applied force*) must be *normal* to the wire (in order to counterbalance the normal component of the magnetic force). We denote the

applied force per unit charge by $\vec{f}_{app}$. Although this force originates from an external mechanical action, it is delivered to *q* through an *electromagnetic* interaction with the crystal lattice of the wire (not to be confused with the resistive force, whose role is different!).

According to (3), the total force contributing to the emf of the circuit is $\vec{f} = \vec{f}_m + \vec{f}_{app}$. By (2), the emf at time *t* is

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}_m \cdot \overrightarrow{dl} + \oint_{C(t)} \vec{f}_{app} \cdot \overrightarrow{dl} \ .$$

The second integral vanishes since the applied force is normal to the wire element at every point of *C*. The integral of the magnetic force is equal to

$$\oint_C (\vec{\upsilon}_{tot} \times \vec{B}) \cdot \overrightarrow{dl} = \oint_C (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

The first integral on the right vanishes, as can be seen by inspecting Fig. 4. Thus, we finally have:

$$\mathcal{E}(t) = \oint_{C(t)} [\vec{\upsilon}(\vec{r}) \times \vec{B}(\vec{r})] \cdot \overrightarrow{dl} \qquad (8)$$

As shown analytically in [1, 2], the emf of *C* is equal to

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) \qquad (9)$$

where we have introduced the *magnetic flux* through *C*,

$$\Phi_m(t) = \int_{S(t)} \vec{B}(\vec{r}) \cdot \overrightarrow{da} = \int_{S(t)} B(\vec{r})\, da \qquad (10)$$

[By *S*(*t*) we denote *any* open surface bounded by *C* at time *t*; e.g., the plane surface enclosed by the wire.]

Now, let C' be the path of *q* in space relative to the external observer, for a full trip of *q* around the wire (in general, C' will be an *open* curve). According to (4), the work done per unit charge for this trip is

$$w = \int_{C'} \vec{f}_m \cdot \overrightarrow{dl'} + \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \ .$$

The first integral vanishes (cf. Fig. 4), while for the second one we notice that

$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl} + \vec{f}_{app} \cdot \overrightarrow{dl''} = \vec{f}_{app} \cdot \overrightarrow{dl''}$$

(since the applied force is normal to the wire element everywhere; see Fig. 4). Thus we finally have:

$$w = \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \qquad (11a)$$

with

$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl''} = \vec{f}_{app} \cdot \vec{\upsilon}\, dt \qquad (11b)$$

where $\overrightarrow{dl''} = \vec{\upsilon}\, dt$ is the infinitesimal displacement of the wire element in time *dt*.

5. An example: Motion inside a uniform magnetic field

Consider a metal bar (*ab*) of length *h*, sliding parallel to itself with constant speed υ on two parallel rails that form part of a U-shaped wire, as shown in Fig. 5. A *uniform* magnetic field $\vec{B}$, pointing into the page, fills the entire region.

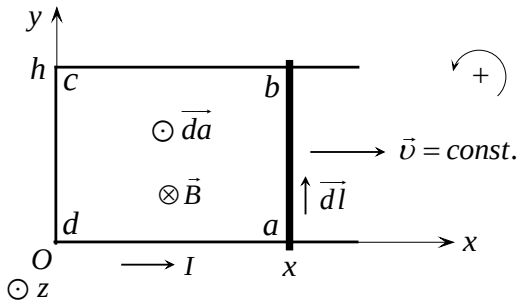

Figure 5: A metal bar (*ab*) sliding on two parallel rails that form part of a U-shaped wire.

A circuit *C*(*t*) of variable size is formed by the rectangular loop (*abcda*). The field and the surface element are written, respectively, as $\vec{B} = -B\hat{u}_z$ (where $B = |\vec{B}| = const.$) and $\overrightarrow{da} = (da)\hat{u}_z$ (note that the direction of traversing the loop *C* is now counterclockwise).

The general diagram of Fig. 4, representing the balance of forces, reduces to the one shown in Fig. 6. Note that this latter diagram concerns only the *moving* part (*ab*) of the circuit, since it is in this part only that the velocity $\vec{\upsilon}$ and the applied force $\vec{f}_{app}$ are nonzero.

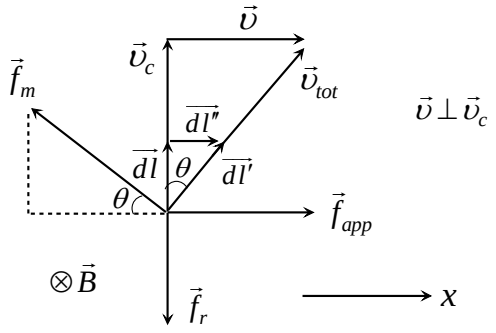

Figure 6: Balance of forces per unit charge.

The emf of the circuit at time *t* is, according to (8),

$$\mathcal{E}(t) = \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl}$$

$$= \int_a^b \upsilon B\, dl = \upsilon B \int_a^b dl = \upsilon B h \ .$$

Alternatively, the magnetic flux through C is

$$\Phi_m(t) = \int_{S(t)} \vec{B}(\vec{r}) \cdot \overrightarrow{da} = -\int_{S(t)} B\, da = -B \int_{S(t)} da$$
$$= -Bhx$$

(where x is the momentary position of the bar at time t), so that

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) = Bh\frac{dx}{dt} = Bh\upsilon \ .$$

We note that the emf is constant (time-independent).

Next, we want to use (11) to evaluate the work per unit charge for a complete tour of a charge around C. Since the applied force is nonzero only on the section (ab) of C, the path of integration, C' (which is a straight line, given that the charge moves at constant velocity in space) will correspond to the motion of the charge along the metal bar only, i.e., from a to b. (Since the bar is being displaced in space while the charge is traveling along it, the line C' will *not* be parallel to the bar.) According to (11),

$$w = \int_{C'} \vec{f}_{app} \cdot \overrightarrow{dl'} \quad \text{with}$$
$$\vec{f}_{app} \cdot \overrightarrow{dl'} = \vec{f}_{app} \cdot \overrightarrow{dl''} = f_{app}\, dl'' = f_{app}\, \upsilon\, dt$$

(cf. Fig. 6). Now, the role of the applied force is to counterbalance the x-component of the magnetic force in order that the bar may move at constant speed in the x direction. Thus,

$$f_{app} = f_m \cos\theta = \upsilon_{tot} B \cos\theta = B\upsilon_c$$

and

$$f_{app}\, \upsilon\, dt = B\upsilon\upsilon_c\, dt = B\upsilon\, dl$$

(since $\upsilon_c\, dt$ represents an elementary displacement dl of the charge along the metal bar in time dt). We finally have:

$$w = \int_a^b B\upsilon\, dl = B\upsilon \int_a^b dl = B\upsilon h \ .$$

We note that, in this specific example, the value of the work per unit charge is equal to that of the emf, both these quantities being constant and unambiguously defined. This would *not* have been the case, however, if the magnetic field were *nonuniform*!

6. Stationary wire inside a time-varying magnetic field

Our final example concerns a *stationary* wire C inside a *time-varying* magnetic field of the form $\vec{B}(\vec{r},t) = -B(\vec{r},t)\,\hat{u}_z$ (where $B(\vec{r},t) = |\,\vec{B}(\vec{r},t)\,|$), as shown in Fig. 7.

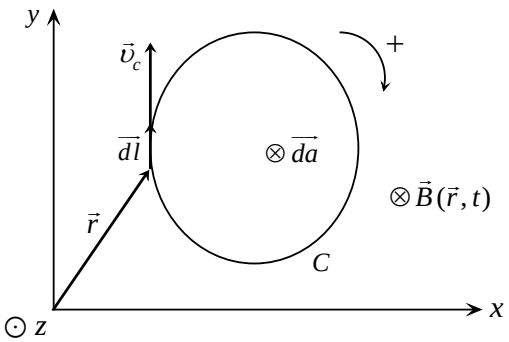

Figure 7: A stationary wire C inside a time-varying magnetic field.

As is well known [1-7], the presence of a time-varying magnetic field implies the presence of an electric field $\vec{E}$ as well, such that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (12)$$

As discussed in [1], the emf of the circuit at time t is given by

$$\mathcal{E}(t) = \oint_C \vec{E}(\vec{r},t) \cdot \vec{dl} = -\frac{d}{dt}\,\Phi_m(t) \qquad (13)$$

where

$$\Phi_m(t) = \int_S \vec{B}(\vec{r},t) \cdot \overrightarrow{da} = \int_S B(\vec{r},t)\, da \qquad (14)$$

is the magnetic flux through C at this time.

On the other hand, the work per unit charge for a full trip around C is given by (5): $w = \oint_C \vec{f} \cdot \overrightarrow{dl}$, where $\vec{f} = \vec{f}_{em} = \vec{E} + (\vec{\upsilon}_c \times \vec{B})$, so that

$$w = \oint_C \vec{E} \cdot \overrightarrow{dl} + \oint_C (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

As is easy to see (cf. Fig. 7), the second integral vanishes, thus we are left with

$$w = \oint_C \vec{E} \cdot \overrightarrow{dl} \qquad (15)$$

The similarity of the integrals in (13) and (15) is deceptive! The integral in (13) is evaluated *at a fixed time t*, while in (15) time is allowed to flow as the charge moves along *C*. Is it, nevertheless, possible that the *values* of these integrals coincide? As mentioned at the end of Sec. 2, a necessary condition for this to be the case is that the two integrations yield time-independent results. In order that $\mathcal{E}$ be time-independent (but nonzero), the magnetic flux (14) – thus the magnetic field itself – must increase *linearly* with time. On the other hand, the integration (15) for *w* will be time-independent if so is the electric field. By (12), then, the magnetic field must be linearly dependent on time, which brings us back to the previous condition.

As an example, assume that the magnetic field is of the form

$$\vec{B} = -B_0 t \hat{u}_z \qquad (B_0 = const.) .$$

A possible solution of (12) for $\vec{E}$ is, in cylindrical coordinates,

$$\vec{E} = \frac{B_0 \rho}{2} \hat{u}_\varphi .$$

[We assume that these solutions are valid in a limited region of space (e.g., in the interior of a solenoid whose axis coincides with the *z*-axis) so that ρ is finite in the region of interest.] Now, consider a circular wire *C* of radius *R*, centered at the origin of the *xy*-plane. Then, given that $\vec{dl} = -(dl)\hat{u}_\varphi$,

$$\mathcal{E} = \oint_C \vec{E} \cdot \vec{dl} = -\frac{B_0 R}{2} \oint_C dl = -B_0 \pi R^2 .$$

Alternatively,

$$\Phi_m = \int_S B da = B_0 \pi R^2 t ,$$

so that $\mathcal{E} = -d\Phi_m / dt = -B_0 \pi R^2$. We anticipate that, due to the time constancy of the electric field, the same result will be found for the work *w* by using (15).

7. Concluding remarks

No single, universally accepted definition of the emf seems to exist in the literature of Electromagnetism. The definition given in this article (as well as in [1]) comes close to those of [2] and [3]. In particular, by using an example similar to that of Sec. 5 in this paper, Griffiths [2] makes a clear distinction between the concepts of emf and work per unit charge. In [4] and [5] (as well as in numerous other textbooks) the emf is identified with work per unit charge, in general, while in [6] and [7] it is defined as a closed line integral of the non-conservative part of the electric field that accompanies a time-varying magnetic flux.

The balance of forces and the origin of work in a conducting circuit moving through a magnetic field are nicely discussed in [2, 8, 9]. An interesting approach to the relation between work and emf, utilizing the concept of virtual work, is described in [10].

Of course, the list of references cited above is by no means exhaustive. It only serves to illustrate the diversity of ideas concerning the concept of the emf. The subtleties inherent in this concept make it an interesting subject of study for both the researcher and the advanced student of classical Electrodynamics.

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Some aspects of the electromotive force: Educational review article

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Synopsis

Certain aspects of the concept of the electromotive force (emf) of a “circuit”, as this concept was defined in recent publications, are discussed. In particular, the independence of the emf from the conductivity of the circuit is explained and the role of the applied force in motional emf is analyzed.

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SOME ASPECTS OF THE
ELECTROMOTIVE FORCE

HELLENIC NAVAL ACADEMY
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Some aspects of the electromotive force

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Certain aspects of the concept of the electromotive force (emf) of a "circuit", as this concept was defined in recent publications, are discussed. In particular, the independence of the emf from the conductivity of the circuit is explained and the role of the applied force in motional emf is analyzed.

1. Definition and analytical expression of the emf

In recent articles [1,2] we studied the concept of the *electromotive force* (*emf*) of a "circuit" and examined the extent to which the emf represents work per unit charge for a complete tour around the circuit. This educational note contains some additional remarks regarding the emf; it may be regarded as an addendum to the aforementioned publications.

We consider a closed path *C* (or *loop*) in a region of space where an electromagnetic (e/m) field exists (Fig. 1). Generally speaking, this loop will be called a *"circuit"* if a charge flow can be sustained on it. We *arbitrarily* assign a positive direction of traversing the loop *C* and we consider an element $\vec{dl}$ of *C* oriented in the positive direction.

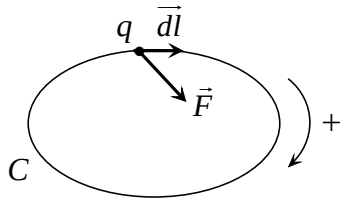

Figure 1

Let *q* be a *test charge*, which at time *t* is located at the position of $\vec{dl}$, and let $\vec{F}$ be the force on *q* at this time. The force $\vec{F}$ is exerted by the e/m field itself as well as, possibly, by additional *energy sources* (such as batteries or some external mechanical action) that may contribute to the generation and preservation of a current around the loop *C*. The *force per unit charge* at the position of $\vec{dl}$, at time *t*, is $\vec{f} = \vec{F}/q$. We note that $\vec{f}$ is independent of *q* since the e/m force on a charge is proportional to the charge.

Since, in general, neither the shape nor the size of *C* is required to remain fixed, and since the loop may also be in motion relative to an external observer, we will use

the notation $C(t)$ to indicate the state, at time t, of a circuit of generally variable shape, size or position in space.

The *electromotive force* (*emf*) of the circuit C *at time* t is defined as the line integral of $\vec{f}$ along C, taken in the *positive* sense of C:

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}(\vec{r},t)\cdot \overrightarrow{dl} \tag{1}$$

where $\vec{r}$ is the position vector of $\overrightarrow{dl}$ relative to the origin of our coordinate system. Obviously, the sign of the emf is dependent upon our choice of the positive direction of circulation of C. It should be noted carefully that the integral (1) is evaluated *at a given time* t. Thus, the force $\vec{f}$ must be measured *simultaneously*, at time t, at all points of C.

The force $\vec{f}$ can be attributed to two factors: (*a*) the interaction of q with the existing e/m field itself; and (*b*) the action on q by any additional energy sources that may be necessary in order to maintain a steady flow of charge on C. (This latter interaction also is *electromagnetic* in nature, even when it originates from some external mechanical action.) We write

$$\vec{f} = \vec{f}_{em} + \vec{f}_{app} \tag{2}$$

where $\vec{f}_{em}$ is the force due to the e/m field and $\vec{f}_{app}$ is the *applied force* due to an additional energy source.

Two familiar cases of emf-driven circuits where an additional applied force is required are the following:

1. In a battery-resistor circuit [1-3] an applied force is necessary in order to carry a (conventionally *positive*) mobile charge from the negative to the positive pole of the battery, *through* the source. This force is provided by the battery itself.

2. In the case of a closed metal wire C moving in a time-independent magnetic field [2-5] the current on C is sustained for as long as the motion of C continues. This, in turn, necessitates the action of an external force on C (say, by our hand), as will be explained in Sec. 4.

Now, by (1) and (2),

$$\mathcal{E}(t) = \oint_{C(t)} \vec{f}_{em}\cdot \overrightarrow{dl} + \oint_{C(t)} \vec{f}_{app}\cdot \overrightarrow{dl} \equiv \mathcal{E}_{em}(t) + \mathcal{E}_{app}(t) \tag{3}$$

We would like to find an analytical expression for $\mathcal{E}_{em}(t)$. So, let $\left(\vec{E}(\vec{r},t), \vec{B}(\vec{r},t)\right)$ be the e/m field in the region of space where the loop $C(t)$ is lying. Let q be a test charge located, at time t, at the position of $\overrightarrow{dl}$ and let $\vec{\upsilon}_{tot}$ be the total velocity of q in space, relative to some inertial frame of reference. We write

$$\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$$

where $\vec{\upsilon}_c$ is the velocity of q along C (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{\upsilon}$ is the velocity of $\overrightarrow{dl}$ itself due to a possible motion in space, or just a deformation over time, of the loop $C(t)$ as a whole. The total e/m force on q is

$$\vec{F}_{em} = q[\vec{E} + (\vec{\upsilon}_{tot} \times \vec{B})] \ ,$$

so that

$$\vec{f}_{em} = \frac{\vec{F}}{q} = \vec{E} + [(\vec{\upsilon} + \vec{\upsilon}_c) \times \vec{B}] \ .$$

Hence,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E} \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} + \oint_{C(t)} (\vec{\upsilon}_c \times \vec{B}) \cdot \overrightarrow{dl} \ .$$

Given that $\vec{\upsilon}_c$ is parallel to $\overrightarrow{dl}$, the last integral on the right vanishes. Thus, finally,

$$\mathcal{E}_{em}(t) = \oint_{C(t)} \vec{E}(\vec{r},t) \cdot \overrightarrow{dl} + \oint_{C(t)} [\vec{\upsilon}(\vec{r},t) \times \vec{B}(\vec{r},t)] \cdot \overrightarrow{dl} \ \equiv \ \mathcal{E}_e(t) + \mathcal{E}_m(t) \qquad (4)$$

We note that, in our definition of the emf, the force per unit charge was defined as $\vec{f} = \vec{F} / q$, assuming that a replica of a test charge q is placed at every point of the circuit and that the forces $\vec{F}$ on all test charges are measured *simultaneously* at time t. Now, in the case of a conducting loop C (say, a metal wire) it is reasonable to identify q with one of the (conventionally positive) mobile free electrons. This particular identification, although logical for practical purposes, is nevertheless not necessary, given that the force $\vec{f}$ is eventually independent of q. Thus, in general, q may just be considered as a *hypothetical* test charge that is not necessarily identified with an actual mobile charge.

2. Independence from conductivity

Let $C(t)$ be a conducting loop (say, a metal wire) inside a given e/m field. The emf of C at time t is given by (3) and (4). We note from (4) that the part $\mathcal{E}_{em}$ of the total emf is independent of the velocity $\vec{\upsilon}_c$ of q *along* C (where q may be conveniently – although not necessarily – assumed to be a mobile free electron of the conductor, conventionally considered as a *positive* charge). We may physically interpret this as follows:

The e/m field creates an emf $\mathcal{E}_{em}$ that tends to generate a charge flow on C. However, this emf does not by itself determine *how fast* the mobile charges move along C. Presumably, this will depend on physical properties of the path C that are associated with its *conductivity*. (For example, in a battery-resistance circuit the potential difference at the ends of the resistance – thus the value of the electric field inside the conductor – does not by itself determine the velocity $\vec{\upsilon}_c$ of the mobile charges along the

circuit, since this velocity is related to the current generated by the source, which current depends, in turn, on the resistance of the circuit, according to Ohm's law.)

Now, the role of the part $\mathcal{E}_{app}$ of the total emf (3) is to *maintain* the charge flow on $C(t)$ that is generated by $\mathcal{E}_{em}$. We thus anticipate that $\mathcal{E}_{app}$ will also be independent of $\vec{\upsilon}_c$ (this is, e.g., the case in our previous example, where $\mathcal{E}_{app}$ is equal to the voltage of the battery [1-3]). In conclusion,

> *the total emf $\mathcal{E}(t)$ of a conducting loop $C(t)$ is not dependent upon the velocity of motion of the mobile charges q along the loop.*

This leads us to a further conclusion:

> *The total emf $\mathcal{E}(t)$ of a conducting loop $C(t)$ inside an e/m field is not dependent upon the conductivity of the loop.*

This can be justified by noting that, by its definition, the force (2) does not include contributions from *resistive forces* that oppose a charge flow on *C*; it only contains e/m interactions that may contribute to the generation and preservation of a current in the circuit. Note, however, that the *current* itself *does* depend on the *conductivity* σ of *C*, according to Ohm's law ($\vec{J} = \sigma \vec{f}$) [3].

Alternatively, as argued above, the emf does not depend on $\vec{\upsilon}_c$. Now, in a steady-state situation under given electrodynamic conditions (thus, for a given $\vec{f}$) this velocity is a linear function of the *mobility* μ of *q*, according to the empirical relation $\vec{\upsilon}_c = \mu \vec{f}$ (by which Ohm's law is deduced). On the other hand, the conductivity of *C* is given by $\sigma=qn\mu$. The *density n* of mobile charges, as well as the value of *q*, cannot affect the value of the emf since that quantity is defined per unit charge. We thus conclude that the emf of *C* cannot depend on μ, as well as on *n* and *q*; hence, $\mathcal{E}$ is independent of σ.

3. Emf and the Faraday-Henry law

Consider a region of space in which a (generally time-dependent) e/m field $(\vec{E}, \vec{B})$ exists. Let *C* be a *fixed* conducting loop in this region. There is no additional applied force on *C*, so (3) reduces to $\mathcal{E}(t)=\mathcal{E}_{em}(t)$. Furthermore, since *C* is stationary, $\vec{\upsilon}(\vec{r},t)$ vanishes identically and, by (4), $\mathcal{E}_m(t)=0$ and $\mathcal{E}_{em}(t)= \mathcal{E}_e(t)$. Thus, finally,

$$\mathcal{E}(t) = \oint_C \vec{E}(\vec{r},t) \cdot \overrightarrow{dl} \tag{5}$$

By Stokes' theorem,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{da}$$

where *S* is any open surface bounded by *C* (Fig. 2).

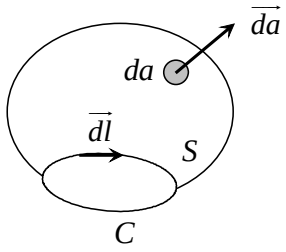

Figure 2

Moreover, by the *Faraday-Henry law*,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

So, (5) yields

$$\mathcal{E}(t) = -\frac{d}{dt}\int_S \vec{B}\cdot\overrightarrow{da} = -\frac{d}{dt}\,\Phi_m(t) \tag{7}$$

where

$$\Phi_m(t) = \int_S \vec{B}(\vec{r},t)\cdot\overrightarrow{da}$$

is the *magnetic flux* through *C* at time *t*. As commented in [1], relation (7) expresses a genuine physical law, not a mere consequence of the definition of the emf.

4. Motional emf due to a static magnetic field

Let *C*(*t*) be a conducting loop inside a static magnetic field $\vec{B}(\vec{r})$ (Fig. 3). The time dependence of *C* indicates a motion and/or a deformation of the loop over time. We will show that the emf of *C* at time *t* is given by the expression

$$\mathcal{E}(t) = \mathcal{E}_m(t) = \oint_{C(t)} [\vec{v}(\vec{r})\times\vec{B}(\vec{r})]\cdot\overrightarrow{dl} \tag{8}$$

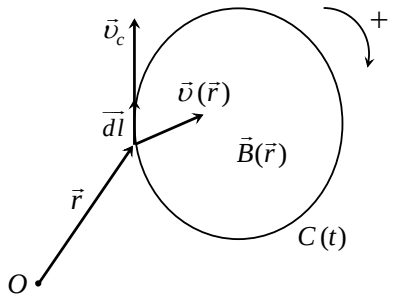

Figure 3

Let q be a mobile charge (say, a conventionally *positive* free electron) located at the position $\vec{r}$ (relative to our coordinate system) of the loop element $\overrightarrow{dl}$ at time t. As in Sec. 1, we denote the velocity of $\overrightarrow{dl}$ with respect to our frame of reference by $\vec{\upsilon}(\vec{r})$, the velocity of q along C by $\vec{\upsilon}_c$, and the total velocity of q relative to our frame by $\vec{\upsilon}_{tot} = \vec{\upsilon} + \vec{\upsilon}_c$.

Since there is no electric field in the region of interest,

$$\mathcal{E}_e(t) \equiv \oint_C \vec{E}(\vec{r},t)\cdot\overrightarrow{dl} = 0 \quad \text{and} \quad \mathcal{E}_{em}(t) = \mathcal{E}_m(t) \tag{9}$$

Also, if $\vec{f}_{app}$ is the applied force per unit charge at the position of q, at time t,

$$\mathcal{E}_{app}(t) = \oint_{C(t)} \vec{f}_{app}(\vec{r},t)\cdot\overrightarrow{dl} \tag{10}$$

The role of the applied force is to keep the current flowing. This will happen for as long as the loop C is moving or/and deforming, so that $\vec{\upsilon}(\vec{r})$ is not identically zero for all t. Why is an external force needed to keep C moving or deforming? Let us carefully analyze the situation.

The magnetic force on q is

$$\vec{F}_m = q(\vec{\upsilon}_{tot}\times\vec{B}) \quad \text{so that} \quad \vec{f}_m = \vec{\upsilon}_{tot}\times\vec{B}\ .$$

Now, imagine a temporary, local 3-dimensional rectangular system of axes (x, y, z) at the location $\vec{r}$ of q at time t. We assume, without loss of generality, that the z-axis is in the direction of $\overrightarrow{dl}$. (The orientation of the mutually perpendicular x and y-axes on the plane normal to the z-axis may be chosen arbitrarily.) Then we may write

$$\vec{f}_m = \vec{f}_{m,x} + \vec{f}_{m,y} + \vec{f}_{m,z} \equiv \vec{f}_c + \vec{f}_\perp$$

where $\vec{f}_c = \vec{f}_{m,z}$ is the component of the magnetic force *along* the loop (i.e., in a direction parallel to $\overrightarrow{dl}$) while $\vec{f}_\perp = \vec{f}_{m,x} + \vec{f}_{m,y}$ is the component *normal* to the loop (thus to $\overrightarrow{dl}$).

In a steady-state situation (steady current flow) $\vec{f}_c$ is counterbalanced by the resistive force that opposes charge motion along C (as mentioned before, this latter force does not contribute to the emf). However, to counterbalance the normal component $\vec{f}_\perp$ some external action (say, by our hand that moves or deforms the loop C) is needed in order for C to keep moving or deforming. This is precisely what the applied force $\vec{f}_{app}$ does. Clearly, this force must be *normal* to C at each point of the loop. From (10) we then conclude that

$$\mathcal{E}_{app}(t) = 0 \, .$$

Combining this with (3), (4) and (9), we finally verify the validity of (8).

It can be shown [1,3] directly from (8) that

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) \tag{11}$$

where $\Phi_m(t)$ is the magnetic flux through C at time t. This *looks like* (7) for a fixed geometrical loop in a time-dependent e/m field, although the origins of the two relations are different. Indeed, equation (11) is a direct consequence of the definition of the emf and may be derived from (8) essentially by mathematical manipulation (see, e.g., the Appendix in [1]). On the contrary, to derive (7) the Faraday-Henry law (6) was used. This is an *experimental* law, hence so is the expression (7) for the emf. In other words, relation (7) is not a mere mathematical consequence of the definition of the emf.

5. An example

Consider a metal bar (*ab*) of length h, sliding parallel to itself with constant speed υ on two parallel rails that form part of a U-shaped wire, as shown in Fig. 4. A *uniform* magnetic field $\vec{B}$, pointing into the page, fills the entire region. A circuit $C(t)$ of variable size is formed by the rectangular loop (*abcda*).

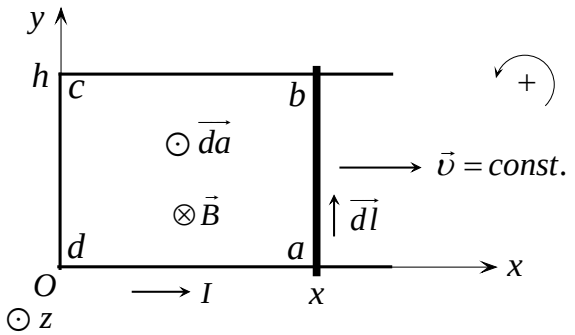

Figure 4

In Fig. 4, the *z*-axis is normal to the plane of the wire and directed toward the reader. We call $\overrightarrow{da}$ an infinitesimal normal vector representing an element of the plane surface bounded by the wire (this vector is directed toward the reader, consistently with the chosen counterclockwise direction of traversing the loop *C*). If $\hat{u}_z$ is the unit vector on the *z*-axis, then the field and the surface element are written, respectively, as $\vec{B} = -B\hat{u}_z$ (where $B = |\vec{B}| = const.$) and $\overrightarrow{da} = (da)\hat{u}_z$.

The balance of forces is shown in Fig. 5 (by $\vec{f}_r$ we denote the resistive force per unit charge, which does not contribute to the emf). Note that this diagram concerns only the *moving* part (*ab*) of the circuit, since it is in this part only that the velocity $\vec{\upsilon}$ and the applied force $\vec{f}_{app}$ are nonzero.

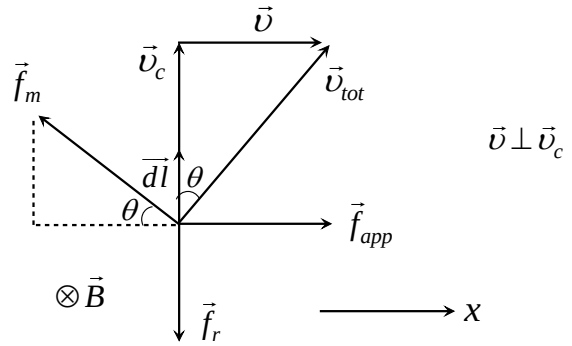

Figure 5

The emf of the circuit at time *t* is, according to (8),

$$\mathcal{E}(t) = \oint_{C(t)} (\vec{\upsilon} \times \vec{B}) \cdot \overrightarrow{dl} = \int_a^b \upsilon B\, dl = \upsilon B \int_a^b dl = \upsilon B h \ .$$

Alternatively, the magnetic flux through *C* is

$$\Phi_m(t) = \int_{S(t)} \vec{B} \cdot \overrightarrow{da} = -\int_{S(t)} B\, da = -B \int_{S(t)} da = -Bhx$$

(where *x* is the momentary position of the bar at time *t*) so that, by (11),

$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_m(t) = Bh\frac{dx}{dt} = Bh\upsilon \ .$$

Now, the role of the applied force is to counterbalance the *x*-component of the magnetic force in order that the bar may move at constant speed in the *x* direction. Thus,

$$f_{app} = f_m \cos\theta = \upsilon_{tot} B \cos\theta = B\upsilon_c \ .$$

We note that, although f_{app} depends on the speed υ_c of a mobile charge along the bar, the associated part of the emf is itself independent of υ_c ! Specifically, as argued in

Sec. 4, $\mathcal{E}_{app}(t)=0$. On the other hand, in this particular example the work w of f_{app} for a complete tour around the circuit is equal to the total emf (cf. [2]): $w=\mathcal{E}=Bh\upsilon$. This equality, however, is accidental and does not reflect a more general relation between the work per unit charge and the emf. (Another such "accidental" case is the battery-resistance circuit [1-3].)

6. Summary

This article is an addendum to our study of the concept of the electromotive force (emf), as this concept was pedagogically approached in previous publications [1,2]. We have focused on some particular aspects of the subject that we felt are important enough to merit further discussion. Let us review them:

1. For a conducting loop C inside an e/m field, we explained why the emf of C does not depend on the conductivity of the loop. As "obvious" as this statement may seem, one still needs to justify it physically and to demonstrate its consistency with Ohm's law.

2. We expressed the Faraday-Henry law in terms of the emf of a closed conducting curve inside a time-dependent e/m field.

3. We studied the case of motional emf in some detail (see also [2-5]). Particularly important is the role of the applied force in this case. In addition to analyzing this role and, in the process, deriving an explicit expression for the emf, we explained why the physics of the situation is different from that of the Faraday-Henry law, despite the similar-looking forms of the emf in the two cases. Of course, as Relativity has shown, this similarity is anything but coincidental!

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Work of a time-dependent force: An addendum to "Does the electromotive force (always) represent work?"

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Synopsis

Certain subtleties concerning the work done by a time-dependent force field are discussed. In particular, it is explained why such a field cannot be conservative even if it is irrotational and its region of action has the proper topological properties.

This article is an addendum to the published article "Does the electromotive force (always) represent work?" (Advanced Electromagnetics, Vol 4, No 1 (2015), pages 10-15, http://www.aemjournal.org/index.php/AEM/article/view/257).

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Work of a time-dependent force[1]

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Abstract

Certain subtleties concerning the work done by a time-dependent force field are discussed. In particular, it is explained why such a field cannot be conservative even if it is irrotational and its region of action has the proper topological properties.

1. Introduction

In a previous article [1] a common misconception regarding the electromotive force (emf) of electrodynamics was discussed. Specifically, it was explained why it is incorrect to *define* the emf as work (per unit charge), in general. In simple terms, the emf is always determined for a given instant of time, whereas in determining the work of a force field on a particle (here, an electric charge) moving along a space curve, time is allowed to flow during the motion. Of course, there *are* exceptional situations where the emf of a circuit does indeed coincide in value with work per unit charge for a complete tour around the circuit [1].

From the point of view of classical mechanics the case of time-dependent forces and their work constitutes an interesting problem. In the present article we highlight certain aspects of this problem, focusing on subtleties that arise when one goes beyond the comfortable case of static force fields. Of course, the subject of time-dependent forces and associated potentials is discussed in many standard textbooks of mechanics (see, e.g., [2-5]). Our aim here is to extend the discussion in these sources by adding a few comments that may help the student to further clarify the situation.

In Section 2 we define the work done by a time-dependent force field on a test particle and point out certain subtle points of this definition.

In Sec. 3 we discuss the relation between irrotational and conservative force fields. We explain why time-dependent fields cannot be conservative and do not lead to conservation of total mechanical energy.

2. Work along a space curve

Consider a test particle of mass m moving in a region of space permeated by a force field $\vec{F}$. The particle is assumed to move along a space curve L extending from point A to point B (Fig. 1). We call $\vec{r}$ the position vector of m on L at time t, relative to the origin O of some inertial reference frame, and we denote by $d\vec{r}$ the elementary displacement of m along L in an infinitesimal time interval dt.

[1] This article is an addendum to the published article [1].

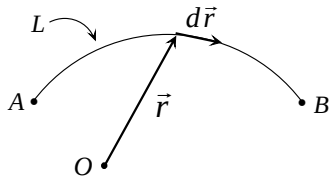

Figure 1

The *work* done by the field $\vec{F}$ on *m* from *A* to *B* is

$$W = \int_L \vec{F} \cdot d\vec{r} \tag{1}$$

To compute the line integral in (1) one needs to have a mathematical description of the curve *L*. Of course, a parametric representation of *L* is possible by using any convenient parameter whose values correspond to the various points $\vec{r}$ of *L*. However, a mere geometrical description of *L* may not be sufficient in order to specify the work *W*, since it may be important to take into account the *time* at which the particle *m* passes through any given point of the curve. Thus, the most faithful parameterization of *L* in this regard is provided by the *equation of motion* of *m*, connecting the position $\vec{r}$ of the particle with the time *t* at which the particle passes from that position.

Let us assume the following mathematical description of the motion of *m* along the trajectory *L*:

$$\vec{r} = \vec{\phi}(t) \;\; ; \;\; t_0 \le t \le t_1 \quad \text{with} \quad \vec{\phi}(t_0) = \vec{r}_A \;, \; \vec{\phi}(t_1) = \vec{r}_B \tag{2}$$

Then, $d\vec{r} = d\vec{\phi}(t) = \vec{\phi}'(t)\,dt$. The complexity of the integration (1) now depends on the nature of the force field $\vec{F}$; specifically, the dependence or not of this field on time.

For a *static* force field $\vec{F}(\vec{r})$, we have:

$$W = \int_{t_0}^{t_1} \vec{F}\left(\vec{\phi}(t)\right) \cdot \vec{\phi}'(t)\,dt \tag{3}$$

This quantity is *independent of the parameterization* of the curve *L*, i.e., independent of the specific functional dependence of $\vec{r}$ on *t* as expressed by (2). Indeed, the substitution $\vec{\phi}(t) = \vec{r}$ transforms the integral (3) into

$$W = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} \tag{4}$$

Evidently, the integral on the right depends only on the geometry of the space curve *L*, not on the specific parameterization of this curve. In conclusion,

> *in a static force field, work is a well-defined quantity depending on the path followed by the particle in the field.*

Things become a lot more complicated in the case of a *time-dependent* force field $\vec{F}(\vec{r},t)$. The work on the particle *m* along the curve *L* is written

$$W = \int_L \vec{F} \cdot d\vec{r} = \int_A^B \vec{F}(\vec{r},t) \cdot d\vec{r} \tag{5}$$

It should be noted carefully that, inside the integral, the variables $\vec{r}$ and t are not independent of each other since the former is a function of the latter through the parameterization (2) of L, i.e., in accordance with the specific equation of motion of m along L. Relation (5) is written

$$W = \int_{t_0}^{t_1} \vec{F}\left(\vec{\phi}\,(t), t\right) \cdot \vec{\phi}\,'(t)\, dt \tag{6}$$

This time the substitution $\vec{\phi}\,(t) = \vec{r}$ will not eliminate t in favor of $\vec{r}$. Thus, the work W is no longer independent of the parameterization of the curve L by the equation of motion of m. The sole geometry of L is not sufficient in order to determine W!

To understand this better, consider the elementary work $dW = \vec{F} \cdot d\vec{r}$. In the case of a static force field, this is written $dW = \vec{F}(\vec{r}) \cdot d\vec{r}$. For a given equation of motion of the form (2), dW depends only *implicitly* on t through the relation $\vec{r} = \vec{\phi}\,(t)$. Thus, for a given elementary displacement of the particle along L, dW depends solely on the position $\vec{r}$ of m on the curve, not on the time at which the particle passes by that position. As t varies from t_0 to t_1, the position vector $\vec{r}$ traces out all curve points from A to B. Eventually, the total work W, given by (4), has a well-defined value independent of the parameterization of L. This work depends only on the geometry of the trajectory L connecting A and B.

On the other hand, in the case of a time-dependent force field the elementary work is of the form $dW = \vec{F}(\vec{r},t) \cdot d\vec{r}$. Here, dW depends *explicitly* on t. Thus, for a given elementary displacement along L, dW depends not only on the position of the particle on L but also on the time the particle passes from that position. This, in turn, depends on the equation of motion $\vec{r} = \vec{\phi}\,(t)$, i.e., on the specific parameterization of L. Therefore the total work (5) is not a uniquely defined quantity but depends on the equation of motion along L.

3. Conservative and irrotational fields

Let $\vec{F}(\vec{r})$ be a static force field. Generally speaking, this field is *conservative* if the work it does on a test particle m is path-independent, or equivalently, if

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0 \tag{7}$$

for any closed path C within the field.

Let S be an open surface bounded by a given closed curve C in the field (Fig. 2). By Stokes' theorem and by Eq. (7),

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{da} = 0 \tag{8}$$

In order for this to be true for every S bounded by C, the field $\vec{F}(\vec{r})$ must be *irrotational*:

$$\vec{\nabla}\times\vec{F}=0 \tag{9}$$

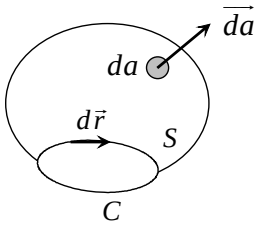

Figure 2

Conversely, an irrotational force field $\vec{F}(\vec{r})$ will also be conservative in a region of space that is *simply connected* [6,7]. Indeed, given any closed curve C in such a region, it is always possible to find an open surface S having C as its boundary. Then, if (9) is valid, the force is conservative in view of (8).

Given a conservative force field $\vec{F}(\vec{r})$, there exists a function $U(\vec{r})$ (*potential energy* of the particle m) such that

$$\vec{F}=-\vec{\nabla}U \tag{10}$$

The work W from point A to point B in the field is then equal to

$$W=\int_A^B \vec{F}(\vec{r})\cdot d\vec{r}=U(\vec{r}_A)-U(\vec{r}_B) \tag{11}$$

As is well known (and as will be shown analytically below) the *total mechanical energy* of m is constant during the particle's motion inside the force field. This energy is the sum $E=T+U$ of the kinetic energy $T=mv^2/2$ (where v is the speed of the particle) and the potential energy U.

Consider now a time-dependent force field $\vec{F}(\vec{r},t)$ in a simply connected region Ω of space. This field is assumed to be irrotational for all values of t:

$$\vec{\nabla}\times\vec{F}(\vec{r},t)=0 \tag{12}$$

Can we conclude that the field $\vec{F}$ is conservative?

It is tempting but *incorrect* (!) to argue as follows: Let C be an arbitrary closed curve in Ω. Since Ω is simply connected, there is always an open surface S bounded by C. By Stokes' theorem,

$$\oint_C \vec{F}(\vec{r},t)\cdot d\vec{r}=\int_S(\vec{\nabla}\times\vec{F})\cdot\overrightarrow{da}=0 \tag{13}$$

for all values of t. This *appears* to imply that $\vec{F}$ is conservative. This is not so, however, for the following reason: For any fixed value of t, the integral

$$I(t) = \oint_C \vec{F}(\vec{r},t)\cdot d\vec{r}$$

does *not* represent work. Indeed, $I(t)$ expresses the integration of a function of two independent variables, $\vec{r}$ and t, over one of these variables (namely, $\vec{r}$), the other variable (t) playing the role of a "parameter" of integration which remains fixed. Thus, $I(t)$ is evaluated *for a given instant of time t* and all values of $\vec{F}$, at the various points of C, must be recorded simultaneously at t.

On the other hand, in the integral representation of work,

$$W = \oint_C \vec{F}(\vec{r},t)\cdot d\vec{r} \ ,$$

time is assumed to flow as the test particle m travels along the closed curve C. In this case, $\vec{r}$ and t are no longer independent of each other but are connected through the equation of motion of m on C, which equation mathematically endows C with a certain parameterization. This complication never arises in the case of static fields, as we saw previously. We may thus conclude that

> *a force field that is both static and irrotational in a simply connected region of space is conservative; a time-dependent force field cannot be conservative even if it is irrotational and its region of action is simply connected.*

Finally, let us explain why a time-dependent force field does not lead to conservation of total mechanical energy. Consider again an irrotational force field $\vec{F}(\vec{r},t)$ [as defined according to (12)] in a simply connected region Ω. Then there exists a time-dependent potential energy $U(\vec{r},t)$ of m, such that, for any value of t,

$$\vec{F}(\vec{r},t) = -\vec{\nabla}U(\vec{r},t) \qquad (14)$$

This time we will assume that $\vec{F}(\vec{r},t)$ is the *total* force on m. By Newton's 2nd law, then,

$$m\frac{d\vec{v}}{dt} = \vec{F} \quad (\text{where } \vec{v} = d\vec{r}/dt) \ \Rightarrow \ m\frac{d\vec{v}}{dt} + \vec{\nabla}U = 0 \ .$$

Taking the dot product with $\vec{v}$, we have:

$$m\vec{v}\cdot\frac{d\vec{v}}{dt} + \vec{v}\cdot\vec{\nabla}U = 0 \ .$$

Now,

$$\vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\frac{d}{dt}(v^2) \quad (v = |\vec{v}|)$$

and

$$\vec{v}\cdot\vec{\nabla}U = \frac{\vec{\nabla}U\cdot d\vec{r}}{dt} = \frac{dU - \dfrac{\partial U}{\partial t}dt}{dt} = \frac{dU}{dt} - \frac{\partial U}{\partial t}$$

where we have used the fact that $dU(\vec{r},t)=\vec{\nabla}U\cdot d\vec{r}+\frac{\partial U}{\partial t}dt$. Hence, finally,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)+\frac{dU}{dt}-\frac{\partial U}{\partial t}=0 \;\Rightarrow$$

$$\frac{d}{dt}(T+U)=\frac{\partial U}{\partial t} \tag{15}$$

where $T=mv^2/2$. As seen from (15), the total mechanical energy ($T+U$) of m is not conserved unless $\partial U/\partial t=0$, i.e., unless the force field is static.

Note that, for a time-dependent irrotational force field [defined according to (12)] the quantity

$$\int_A^B \vec{F}(\vec{r},t)\cdot d\vec{r}=U(\vec{r}_A,t)-U(\vec{r}_B,t)\,,$$

defined for any *fixed t*, does *not* represent the work done by this field on a particle m from A to B [comp. (11) for the case of a static force field]. That is,

> *the work of a time-dependent irrotational force field cannot be expressed as the (negative) difference of the values of the corresponding time-dependent potential energy at the end points of the trajectory of a particle.*

4. Summary

Let us summarize our main conclusions:

1. In a static force field, the work done on a test particle is a well-defined quantity that depends on the geometrical characteristics of the particle's trajectory in the field.

2. In a time-dependent force field, the geometry of the trajectory is not sufficient in order to determine work: one must also know the precise equation of motion of the particle along this trajectory, connecting the position of the particle with time. Thus, work is not a uniquely defined quantity in this case.

3. A static force field that is irrotational in a simply connected region of space is conservative.

4. A time-dependent force field cannot be conservative even if it is irrotational and its region of action has the proper topology.

5. The work of a time-dependent irrotational force field cannot be expressed as the difference of the values of the time-dependent potential energy at the end points of the trajectory of a particle.

6. Time-dependent force fields are incompatible with conservation of total mechanical energy.

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Revisiting the Charging–Capacitor Problem: Maxwell's Equations and Approximate Solutions

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Abstract. The charging capacitor is used as a standard paradigm for illustrating the concept of the Maxwell "displacement current". A certain aspect of the problem, however, is often overlooked. It concerns the conditions for satisfaction of the Faraday-Henry law both in the interior and in the exterior of the capacitor. In this article the situation is analyzed and a recursive process is described for obtaining (at least approximate) solutions of Maxwell's equations inside and outside the capacitor.

Keywords: Charging capacitor, Maxwell equations, Faraday-Henry law

Introduction

The charging capacitor is used as a standard paradigm for demonstrating the significance of the Maxwell "displacement current" (see, e.g., Griffiths, 2013; Wangsness, 1986; Shadowitz, 1975; Rojansky, 1979; Jackson, 1999; McDonald, 2017; Selvan, 2009). The point is correctly made that, without this "current" term the static Ampère's law would be incomplete with regard to explaining the conservation of charge as well as the existence of electromagnetic radiation. Furthermore, the line integral of the magnetic field around a closed curve would be an ill-defined concept (see Appendix II).

A certain aspect of the problem, however, is often overlooked in the educational literature (Papachristou, 2018; Papachristou and Magoulas, 2018). It concerns the satisfaction of the Faraday-Henry law both inside and outside the capacitor. Indeed, although care is taken to ensure that the expressions used for the electromagnetic (e/m) field satisfy the Ampère-Maxwell law, no such care is exercised with regard to the Faraday-Henry law. As it turns out, the usual formulas for the e/m field satisfy this latter law only in the special case where the capacitor is being charged at a constant rate. But, if the current responsible for charging the capacitor is time-dependent, this will also be the case with the magnetic field outside the capacitor. This, in turn, implies the existence of an "induced" electric field in that region, contrary to the usual assertion that the electric field outside the capacitor is zero. Moreover, the time dependence of the magnetic field inside the capacitor is not compatible with the assumption that the electric field in that region is uniform, as the case would be in a static situation. Thus, the expressions usually given in the literature for the e/m field inside and outside a charging capacitor fail to satisfy the Faraday-Henry law in the case of a time-dependent current.

In this article we describe a method for finding expressions for the e/m field that properly satisfy the full set of Maxwell's equations (including, of course, the Faraday-Henry law) both inside and outside the capacitor. These solutions depend on two scalar functions of space and time, which functions satisfy a certain system of partial differential equations (PDEs). The time-dependent current that charges the capacitor appears as a sort of parametric function in this system.

We suggest a mathematical process for obtaining solutions of the above-mentioned system of PDEs in the form of power series with respect to time. This allows one to find approximate expressions for the e/m field in certain situations. For example, a slowly varying (thus almost time-independent) current allows for the "classical" (albeit incorrect in precise terms) solutions given in the literature, while a current that is almost linearly dependent on time (as may be assumed, in general, for any smoothly varying current in a very short time period) allows for new solutions that correct the standard expressions for the electric field while retaining the corresponding expressions for the magnetic field.

It should be noted that, regarding the solutions in the exterior of the capacitor, no retardation effects related to the finite speed of propagation of e/m interactions will concern us here. As discussed at the end of the article, our solutions are valid at points of space not far from the capacitor, so that any change in the physical system will practically be felt simultaneously at all points of interest.

Solutions of Maxwell's equations inside the capacitor

We consider a parallel-plate capacitor with circular plates of radius a, thus of area $A=\pi a^2$. The space in between the plates is assumed to be empty of matter. The capacitor is being charged by a time-dependent current $I(t)$ flowing in the $+z$ direction (see Fig. 1). The z-axis is perpendicular to the plates (the latter are therefore parallel to the xy-plane) and passes through their centers, as seen in the figure (by $\hat{u}_z$ we denote the unit vector in the $+z$ direction).

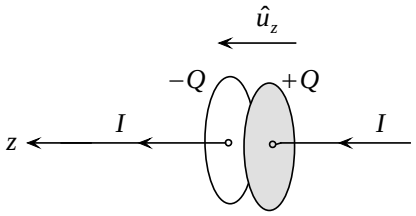

Figure 1. A parallel-plate capacitor charged by a current $I(t)$.

The capacitor is being charged at a rate $dQ/dt=I(t)$, where $+Q(t)$ is the charge on the right plate (as seen in the figure) at time t. If $\sigma(t)=Q(t)/\pi a^2=Q(t)/A$ is the surface charge density on the right plate, then the time derivative of σ is given by

$$\sigma'(t)=\frac{Q'(t)}{A}=\frac{I(t)}{A} \tag{1}$$

We assume that the plate separation is very small compared to the radius a, so that the e/m field inside the capacitor is practically independent of z, although it *does* depend on the normal distance ρ from the z-axis. In cylindrical coordinates (ρ,φ,z) the magnitude of the e/m field at any time t will thus only depend on ρ (due to the symmetry of the problem, this magnitude will not depend on the angle φ).

We assume that the positive and the negative plate of the capacitor of Fig. 1 are centered at $z=0$ and $z=d$, respectively, on the z-axis, where, as mentioned above, the plate separation d is much smaller than the radius a of the plates. The interior of the capacitor is then the region of space with $0\le\rho<a$ and $0<z<d$.

The magnetic field inside the capacitor is azimuthal, of the form $\vec{B} = B(\rho,t)\hat{u}_\varphi$. A standard practice in the literature is to assume that, at all *t*, the electric field in this region is uniform, of the form

$$\vec{E} = \frac{\sigma(t)}{\varepsilon_0}\,\hat{u}_z \tag{2}$$

while everywhere outside the capacitor the electric field vanishes. With this assumption the magnetic field inside the capacitor is found to be (Wangsness, 1986; Shadowitz, 1975; McDonald, 2017)

$$\vec{B} = \frac{\mu_0 I(t)\rho}{2\pi a^2}\,\hat{u}_\varphi = \frac{\mu_0 I(t)\rho}{2A}\,\hat{u}_\varphi \tag{3}$$

Expressions (2) and (3) must, of course, satisfy the Maxwell system of equations in empty space, which system we choose to write in the form (Griffiths, 2013; Papachristou, 2020)

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E} = 0 \qquad && (c)\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B} = 0 \qquad && (d)\quad \vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\,\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{4}$$

By using cylindrical coordinates (see Appendix I) and by taking (1) into account, one may show that (2) and (3) satisfy three of Eqs. (4), namely, (*a*), (*b*) and (*d*). This is not the case with the Faraday-Henry law (4*c*), however, since by (2) and (3) we find that $\vec{\nabla}\times\vec{E} = 0$ while

$$\frac{\partial\vec{B}}{\partial t} = \frac{\mu_0 I'(t)\rho}{2A}\,\hat{u}_\varphi\ .$$

An exception occurs if the current *I* is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$. This is actually the assumption silently or explicitly made in many textbooks [see, e.g., Wangsness (1986), Chap. 21]. But, for a current $I(t)$ with arbitrary time dependence, the pair of fields (2) and (3) does not satisfy the third Maxwell equation.

To remedy the situation and restore the validity of the full set of Maxwell's equations in the interior of the capacitor, we must somehow correct the above expressions for the e/m field. To this end we employ the following *Ansatz*, taking into account Lemma 1 in Appendix III:

$$\vec{E} = \left(\frac{\sigma(t)}{\varepsilon_0} + f(\rho,t) \right) \hat{u}_z \ ,$$
$$\vec{B} = \left(\frac{\mu_0 I(t)\rho}{2A} + g(\rho,t) \right) \hat{u}_\varphi \ ; \qquad (5)$$
$$\sigma'(t) = I(t)/A$$

where $f(\rho,t)$ and $g(\rho,t)$ are functions to be determined consistently with the given current function $I(t)$ and the given initial conditions. It can be checked that the solutions (5) automatically satisfy the first two Maxwell equations (4*a*) and (4*b*). By the Faraday-Henry law (4*c*) and the Ampère-Maxwell law (4*d*) we get the following system of PDEs:

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)\rho}{2A}$$
$$\frac{1}{\rho}\frac{\partial(\rho g)}{\partial \rho} = \varepsilon_0 \mu_0 \frac{\partial f}{\partial t} \qquad (6)$$

Note in particular that the "classical" solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the current I is constant in time, which means that the capacitor is being charged at a constant rate.

The quantity $(1/\rho)\partial(\rho g)/\partial\rho$ in the second equation, having its origin at the expression for $\vec{\nabla}\times\vec{B}$ in cylindrical coordinates, must tend to a finite limit for $\rho\to 0$ in order that the *rot* of the magnetic field be finite at the center of the capacitor. For this to be the case, $\partial(\rho g)/\partial\rho$ must only contain terms of at least first order in ρ. This, in turn, requires that g itself must be of at least first order (i.e., linear with no constant term) in ρ for all t, or else g must be identically zero. We must, therefore, require that

$$g(\rho,t) \to 0 \ \text{ for } \ \rho\to 0 \qquad (7)$$

for all t. Keeping this condition in mind, we can rewrite the system (6) in a more symmetric form:

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)\rho}{2A}$$
$$\frac{\partial(\rho g)}{\partial \rho} = \varepsilon_0 \mu_0 \frac{\partial(\rho f)}{\partial t} \qquad (8)$$

In principle, one needs to solve the system (8) for a given current $I(t)$ and for given initial conditions. An alternative approach, leading to approximate solutions of various forms, is to expand all functions (i.e., f, g and I) in powers of time, t. We thus write:

$$I(t) = \sum_{n=0}^{\infty} I_n t^n \qquad (9a)$$

$$f(\rho,t)=\sum_{n=0}^{\infty} f_n(\rho)\,t^n \tag{9b}$$

$$g(\rho,t)=\sum_{n=0}^{\infty} g_n(\rho)\,t^n \tag{9c}$$

Then, for example,

$$I'(t)=\sum_{n=1}^{\infty} nI_n\,t^{n-1}=\sum_{n=0}^{\infty}(n+1)I_{n+1}\,t^n\ ,\ \text{etc.}$$

Obviously, I_n has dimensions of current × (time)$^{-n}$, while f_n and g_n have dimensions of field intensity (electric and magnetic, respectively) × (time)$^{-n}$.

Substituting the series expansions (9) into the system (8), and equating coefficients of similar powers of t on both sides of the ensuing equations, we get a recursion relation in the form of a system of PDEs:

$$\begin{aligned} f_n'(\rho)&=(n+1)\left[g_{n+1}(\rho)+\frac{\mu_0\rho}{2A}I_{n+1}\right]\\ \left[\rho g_n(\rho)\right]'&=(n+1)\,\varepsilon_0\mu_0\,\rho\,f_{n+1}(\rho)\end{aligned} \tag{10}$$

for n=0,1,2,... All non-vanishing functions $g_n(\rho)$ are required to satisfy the boundary condition (7); i.e., $g_n(\rho)\to 0$ for $\rho\to 0$.

An obvious solution of the system (10) is the trivial solution $f_n(\rho)\equiv 0$ and $g_n(\rho)\equiv 0$ for all n=0,1,2,..., corresponding to $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$. For this to be the case, we must have $I_{n+1}=0$ for all n=0,1,2,..., which means that $I(t)=I_0$=constant (independent of t). This is the case typically treated in the literature, although the condition I=*const.* is usually not stated explicitly.

The simplest nontrivial solution of the problem is found by assuming that f and g are time-independent, i.e., are functions of ρ only. Then, by (9*b*) and (9*c*), $f=f_0(\rho)$ and $g=g_0(\rho)$, while $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>0$. The system (10) for n=0 gives

$$f_0'(\rho)=\frac{\mu_0 I_1\rho}{2A}\quad\text{and}\quad\left[\rho g_0(\rho)\right]'=0$$

with solutions

$$f_0(\rho)=\frac{\mu_0 I_1\rho^2}{4A}+C\quad\text{and}\quad g_0(\rho)=\frac{\lambda}{\rho}\ ,$$

respectively. The boundary condition $g_0(\rho)\to 0$ for $\rho\to 0$ cannot be satisfied for $\lambda\neq 0$; we are thus compelled to set λ=0. Given that $f(\rho,t)=f_0(\rho)$ and $g(\rho,t)=g_0(\rho)$, the solution of the system (8) is

$$f(\rho,t)=\frac{\mu_0 I_1\rho^2}{4A}+C\ ,\quad g(\rho,t)\equiv 0 \tag{11}$$

As is easy to check, by the first of Eqs. (10) it follows that I_n=0 for n>1. Therefore $I(t)$ is linear in t, i.e., is of the form $I(t)=I_0+I_1 t$. By assuming the initial condition $I(0)=0$, we have that $I_0=0$ and

$$I(t) = I_1\, t \tag{12}$$

On the other hand, by integrating Eq. (1): $\sigma'(t)=I(t)/A$, and by assuming that the capacitor is initially uncharged [$\sigma(0)=0$], we get:

$$\sigma(t) = \frac{I_1 t^2}{2A} \tag{13}$$

Finally, by Eqs. (5), (11), (12) and (13) the e/m field in the interior of the capacitor is

$$\begin{aligned} \vec{E} &= \left(\frac{I_1 t^2}{2\varepsilon_0 A} + \frac{\mu_0 I_1 \rho^2}{4A} \right) \hat{u}_z \ , \\ \vec{B} &= \frac{\mu_0 I_1 t \rho}{2A}\, \hat{u}_\varphi \end{aligned} \tag{14}$$

where we have set C=0 since, in view of the assumed initial conditions, there is no electric field inside the capacitor if I_1=0. In order for the solution (14) to be valid, the current $I(t)$ charging the capacitor must vary linearly with time, according to (12).

Solutions of Maxwell's equations outside the capacitor

We recall that the positive and the negative plate of the capacitor of Fig. 1 are centered at z=0 and z=d, respectively, on the z-axis, where the plate separation d is much smaller than the radius a of the plates. The space exterior to the capacitor consists of points with $\rho > 0$ and $z \notin (0,d)$, as well as points with $\rho > a$ and $0 < z < d$. (In the former case we exclude points on the z-axis, with ρ=0, to ensure the finiteness of our solutions in that region.) We assume that the current $I(t)$ is of "infinite" extent and hence the magnitude of the e/m field is practically z-independent.

The e/m field outside the capacitor is usually described mathematically by the equations (Wangsness, 1986; Shadowitz, 1975; McDonald, 2017)

$$\vec{E} = 0 \ , \quad \vec{B} = \frac{\mu_0 I(t)}{2\pi\rho}\, \hat{u}_\varphi \tag{15}$$

As the case is with the standard solutions in the interior of the capacitor, the solutions (15) fail to satisfy the Faraday-Henry law (4c) (although they do satisfy the remaining three Maxwell equations), since $\vec{\nabla} \times \vec{E} = 0$ while

$$\frac{\partial \vec{B}}{\partial t} = \frac{\mu_0 I'(t)}{2\pi\rho}\, \hat{u}_\varphi \ .$$

As before, an exception occurs if the current I is constant in time, i.e., if the capacitor is being charged at a constant rate, so that $I'(t)=0$.

To find more general solutions that satisfy the entire set of the Maxwell equations, we work as in the previous section. Taking into account Lemma 2 in Appendix III, we assume the following general form of the e/m field everywhere outside the capacitor:

$$\begin{aligned} \vec{E} &= f(\rho,t)\,\hat{u}_z\ , \\ \vec{B} &= \left(\frac{\mu_0 I(t)}{2\pi\rho} + g(\rho,t)\right)\hat{u}_\varphi \end{aligned} \tag{16}$$

where f and g are functions to be determined consistently with the given current function $I(t)$. The solutions (16) automatically satisfy the first two Maxwell equations (4*a*) and (4*b*). By Eqs. (4*c*) and (4*d*) we get the following system of PDEs:

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{\partial g}{\partial t} + \frac{\mu_0 I'(t)}{2\pi\rho} \\ \frac{\partial(\rho g)}{\partial \rho} &= \varepsilon_0\mu_0 \frac{\partial(\rho f)}{\partial t} \end{aligned} \tag{17}$$

Again, the usual solution with $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$ is possible only if $I'(t)=0$, i.e., if the capacitor is being charged at a constant rate. Note also that, since now $\rho\neq 0$, the boundary condition (7) for g no longer applies.

As we did in the previous section, we seek a series solution of the system (17) in powers of t. We thus expand f, g and I as in Eqs. (9), substitute the expansions into the system (17), and compare terms with equal powers of t. The result is a new recursive system of PDEs:

$$\begin{aligned} f_n'(\rho) &= (n+1)\left[\, g_{n+1}(\rho) + \frac{\mu_0}{2\pi\rho} I_{n+1} \,\right] \\ \left[\rho g_n(\rho)\right]' &= (n+1)\,\varepsilon_0\mu_0\,\rho\, f_{n+1}(\rho) \end{aligned} \tag{18}$$

for $n=0,1,2,...$ Again, an obvious solution is the trivial solution $f_n(\rho)\equiv 0$ and $g_n(\rho)\equiv 0$ for all $n=0,1,2,...$, corresponding to $f(\rho,t)\equiv 0$ and $g(\rho,t)\equiv 0$. This requires that $I_{n+1}=0$ for all $n=0,1,2,...$, so that $I(t)=I_0=$constant (independent of t).

As we did previously, we seek time-independent solutions for f and g, so that $f=f_0(\rho)$ and $g=g_0(\rho)$ while $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>0$. The system (18) for $n=0$ gives

$$f_0'(\rho) = \frac{\mu_0 I_1}{2\pi\rho} \quad \text{and} \quad \left[\rho g_0(\rho)\right]' = 0$$

with solutions

$$f_0(\rho) = \frac{\mu_0 I_1}{2\pi}\ln(\kappa\rho) \quad \text{and} \quad g_0(\rho) = \frac{\lambda}{2\pi\rho}\ ,$$

respectively (remember that $\rho>0$), where κ is a positive constant quantity having dimensions of inverse length, and where a factor of 2π has been put in $g_0(\rho)$ for future convenience. Given that $f(\rho,t)=f_0(\rho)$ and $g(\rho,t)=g_0(\rho)$, the solution of the system (17) is

$$f(\rho,t)=\frac{\mu_0 I_1}{2\pi}\ln(\kappa\rho)\ , \quad g(\rho,t)=\frac{\lambda}{2\pi\rho} \tag{19}$$

By the first of Eqs. (18) it follows that $I_n=0$ for $n>1$. Therefore $I(t)$ is linear in t, of the form $I(t)=I_0+I_1 t$. By assuming the initial condition $I(0)=0$, we have that $I_0=0$ and

$$I(t) = I_1\, t \tag{20}$$

In view of the above results, the e/m field (16) in the exterior of the capacitor is

$$\begin{aligned} \vec{E} &= \frac{\mu_0 I_1}{2\pi}\ln(\kappa\rho)\,\hat{u}_z\ , \\ \vec{B} &= \frac{\mu_0 I_1 t+\lambda}{2\pi\rho}\,\hat{u}_\varphi \end{aligned} \tag{21}$$

For this solution to be valid, the current $I(t)$ must vary linearly with time.

By comparing Eqs. (14) and (21) we observe that the value of the electric field inside the capacitor does not match the value of this field outside for $\rho=a$, where a is the radius of the capacitor. This discontinuity of the electric field at the boundary of the space occupied by the capacitor is a typical characteristic of capacitor problems, in general. On the other hand, in order that the magnetic field in the strip $0 < z < d$ be continuous for $\rho=a$, the expression for $\vec{B}$ in (21) must match the corresponding expression in (14) upon substituting $\rho=a$ and by taking into account that $A=\pi a^2$. This requires that we set $\lambda=0$ in (21), so that this equation finally becomes

$$\begin{aligned} \vec{E} &= \frac{\mu_0 I_1}{2\pi}\ln(\kappa\rho)\,\hat{u}_z\ , \\ \vec{B} &= \frac{\mu_0 I_1 t}{2\pi\rho}\,\hat{u}_\varphi \end{aligned} \tag{22}$$

Discussion

As we have seen, expressions for the e/m field inside and outside a charging capacitor may be sought in the general form given by Eqs. (5) and (16), respectively. These expressions contain two unknown functions $f(\rho,t)$ and $g(\rho,t)$ which, in view of Maxwell's equations, satisfy the systems of PDEs (8) and (17). These PDEs, in turn, admit series solutions in powers of t, of the form (9), where it is assumed that the current $I(t)$ itself may be expanded in this fashion.

The coefficients of expansion of f and g may be determined, in principle, by means of the recursion relations (10) and (18), both of which are of the general form

$$f_n'(\rho) = (n+1)\left[g_{n+1}(\rho) + h(\rho) I_{n+1}\right]$$
$$\left[\rho g_n(\rho)\right]' = (n+1)\,\varepsilon_0 \mu_0\, \rho\, f_{n+1}(\rho) \qquad (23)$$

This is not an easy system to integrate, so we are compelled to make certain *ad hoc* assumptions. Suppose, e.g., that we seek a solution such that $f_n(\rho)=0$ and $g_n(\rho)=0$ for $n>k$ ($k\geq 0$). It then follows from the first of Eqs. (23) that $I_{n+1}=0$ for $n>k$ or, equivalently, $I_n=0$ for $n>k+1$. Thus, if $k=0$, $I(t)$ must be linear in t; if $k=1$, $I(t)$ must be quadratic in t; etc.

For a current varying sufficiently slowly with time, we may approximately assume that $I_n=0$ for $n>0$, so that $I(t)=I_0=const$. This allows for the possibility that f and g vanish identically, as is effectively assumed (though not always stated explicitly) in the literature [see, however, Milsom (2020)]. On the other hand, any smoothly varying $I(t)$ may be assumed to vary linearly with time for a very short time period. Then, a solution of the form (14) and (22) is admissible.

There are several aspects of the solutions described by Eqs. (14) and (22) that may look unphysical: (*a*) the electric field in (22) apparently diverges for $\rho\to\infty$; (*b*) the magnetic field in both (14) and (22) diverges for $t\to\infty$; (*c*) although, by assumption, there are no charges at the interface between the interior and the exterior of the capacitor (i.e., on the cylindrical surface defined by $0< z < d$ and $\rho=a$) the electric field is non-continuous on that surface, contrary to the general boundary conditions required by Maxwell's equations; (*d*) the constant κ in (22) appears to be arbitrary. We may thus use the above solutions only as approximate ones for values of ρ not much larger than the radius a of the plates, as well as for short time intervals. (Note that ρ has to be much smaller than the length of the wire that charges the capacitor if this wire is to be considered of "infinite" length, hence if the external e/m field is to be regarded as z-independent.) We may smoothen the discontinuity problem of the electric field for $\rho=a$ by assuming that this field is continuous at $t=0$, i.e., at the moment when the charging of the capacitor begins. By setting $\rho=a$ in (14) and (22) and by equating the corresponding expressions for $\vec{E}$ we may then determine the value of the constant κ in (22). The result is: $\kappa=e^{1/2}/a$.

For an enlightening discussion of the subtleties concerning the e/m field produced by an infinitely long straight current, the reader is referred to Example 7.9 of Griffiths (2013).

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I thank Aristidis N. Magoulas for a number of helpful discussions on this subject.

Appendix I. Vector operators in cylindrical coordinates

Let $\vec{A}$ be a vector field, expressed in cylindrical coordinates (ρ, φ, z) as

$$\vec{A} = A_\rho(\rho,\varphi,z)\,\hat{u}_\rho + A_\varphi(\rho,\varphi,z)\,\hat{u}_\varphi + A_z(\rho,\varphi,z)\,\hat{u}_z \ .$$

The *div* and the *rot* of this field in this system of coordinates are written, respectively, as follows:

$$\vec{\nabla}\cdot\vec{A}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}\ ,$$

$$\vec{\nabla}\times\vec{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi}-\frac{\partial A_\varphi}{\partial z}\right)\hat{u}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{u}_\varphi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\varphi)-\frac{\partial A_\rho}{\partial\varphi}\right)\hat{u}_z\ .$$

In particular, if the vector field is of the form

$$\vec{A}=A_\varphi(\rho)\,\hat{u}_\varphi+A_z(\rho)\,\hat{u}_z\ ,$$

then $\vec{\nabla}\cdot\vec{A}=0$.

Appendix II. Charging capacitor: The "textbook" approach

When writing the Ampère-Maxwell law in its integral form, one must carefully define the concept of the *total current through a loop C* (where by "loop" we mean a closed curve in space).

Proposition. Consider a region *R* of space within which the distribution of charge, expressed by the volume charge density, is time-independent. Let *C* be an oriented loop in *R*, and let *S* be any open surface in *R* bordered by *C* and oriented accordingly. We define the total current through *C* as the surface integral of the current density $\vec{J}$ over *S*:

$$I_{in}=\int_S \vec{J}\cdot\overrightarrow{da} \qquad \text{(A.1)}$$

Then, the quantity I_{in} has a well-defined value independent of the particular choice of *S* (that is, I_{in} is the same for all open surfaces *S* bounded by *C*).

Proof. By the equation of continuity for the electric charge [see, e.g., Papachristou (2020), Chap. 6] and by the fact that the charge density inside the region *R* is static, we have that $\vec{\nabla}\cdot\vec{J}=0$. Therefore, within this region of space the current density has the properties of a solenoidal field. In particular, the value of the surface integral of $\vec{J}$ will be the same for all open surfaces *S* sharing a common border *C*.

As an example, let us consider a circuit carrying a time-dependent current *I*(*t*). If the circuit does not contain a capacitor, no charge is piling up at any point and the charge density at any elementary segment of the circuit is constant in time. Moreover, at each instant *t*, the current *I* is constant along the circuit, its value changing only with time. Now, if *C* is a loop encircling some section of the circuit, as shown in Fig. 2, then, at each instant *t*, the same current *I*(*t*) will pass through any open surface *S* bordered by *C*. Thus, the integral in (A.1) is well defined for all *t*, assuming the same value $I_{in}=I(t)$ for all *S*.

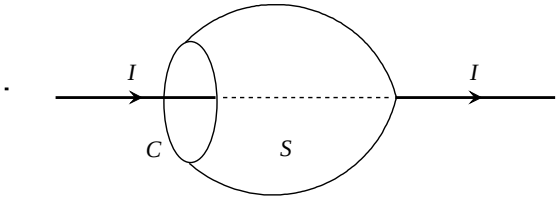

Figure 2. An open surface *S* bordered by a closed curve *C* encircling a current *I*.

Things change if the circuit contains a capacitor that is charging or discharging. It is then no longer true that the current $I(t)$ is constant along the circuit; indeed, $I(t)$ is zero inside the capacitor and nonzero outside. Thus, the value of the integral in (A.1) depends on whether the surface *S* does or does not contain points belonging to the interior of the capacitor.

Figure 3 shows a simple circuit containing a capacitor that is being charged by a time-dependent current $I(t)$. At time *t*, the plates of the capacitor, each of area *A*, carry charges $\pm Q(t)$.

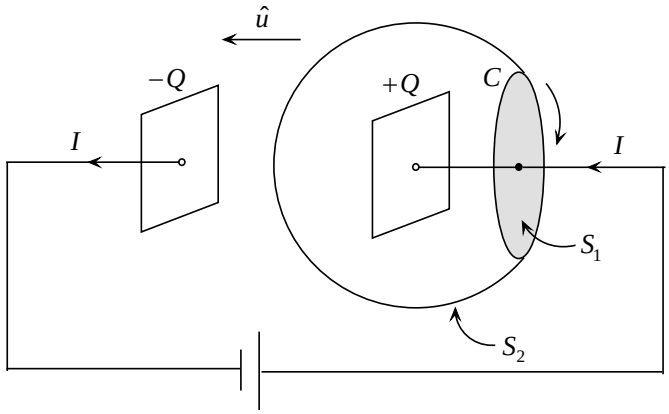

Figure 3. A circuit containing a capacitor charged by a current $I(t)$.

Assume that we encircle the current *I* by an imaginary plane loop *C* parallel to the positive plate and oriented in accordance with the "right-hand rule", consistently with the direction of *I* (this direction is indicated by the unit vector $\hat{u}$). The "current through *C*" is here an ill-defined notion since the value of the integral in Eq. (A.1) is $I_{in}=I$ for the flat surface S_1 and $I_{in}=0$ for the curved surface S_2. This, in turn, implies that Ampère's law of magnetostatics [see, e.g., Papachristou (2020), Chap. 7] cannot be valid in this case, given that, according to this law, the integral of the magnetic field $\vec{B}$ along the loop *C*, equal to $\mu_0 I_{in}$, would not be uniquely defined but would depend on the choice of the surface *S* bounded by *C*.

Maxwell restored the single-valuedness of the closed line integral of $\vec{B}$ by introducing the so-called *displacement current*, which is essentially the rate of change of a time-dependent electric field:

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Leftrightarrow \quad I_d = \int_S \vec{J}_d \cdot \overrightarrow{da} = \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{da} \tag{A.2}$$

The *Ampère-Maxwell law* reads:

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\quad\Leftrightarrow$$
$$\oint_C\vec{B}\cdot\overrightarrow{dl}=\mu_0 I_{in}+\varepsilon_0\mu_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot\overrightarrow{da}\equiv\mu_0\,(I+I_d)_{in} \tag{A.3}$$

where I_{in} is given by Eq. (A.1).

Now, the standard "textbook" approach to the charging capacitor problem goes as follows: Outside the capacitor the electric field vanishes everywhere, while inside the capacitor the electric field is uniform – albeit time-dependent – and has the static-field-like form

$$\vec{E}=\frac{\sigma(t)}{\varepsilon_0}\,\hat{u}\;=\frac{Q(t)}{\varepsilon_0 A}\,\hat{u} \tag{A.4}$$

where $\sigma(t)=Q(t)/A$ is the surface charge density on the positive plate at time t. This density is related to the current I that charges the capacitor by

$$\sigma'(t)=\frac{Q'(t)}{A}=\frac{I(t)}{A} \tag{A.5}$$

(the prime indicates differentiation with respect to t). Thus, inside the capacitor,

$$\frac{\partial\vec{E}}{\partial t}=\frac{\sigma'(t)}{\varepsilon_0}\,\hat{u}\;=\frac{I(t)}{\varepsilon_0 A}\,\hat{u} \tag{A.6}$$

Outside the capacitor the time derivative of the electric field vanishes everywhere and, therefore, so does the displacement current.

Now, on the flat surface S_1 the total current through C is $(I+I_d)_{in}=I+0=I(t)$. The Ampère-Maxwell law (A.3) then yields:

$$\int_C\vec{B}\cdot\overrightarrow{dl}=\mu_0 I(t) \tag{A.7}$$

On the curved surface S_2 the total current through C is $(I+I_d)_{in}=0+I_{d,in}=I_{d,in}$, where the quantity on the right assumes a nonzero value only for the portion S_2' of S_2 that lies inside the capacitor. This quantity is equal to

$$I_{d,in}=\varepsilon_0\int_{S_2'}\frac{\partial\vec{E}}{\partial t}\cdot\overrightarrow{da}=\frac{I(t)}{A}\int_{S_2'}\hat{u}\cdot\overrightarrow{da} \tag{A.8}$$

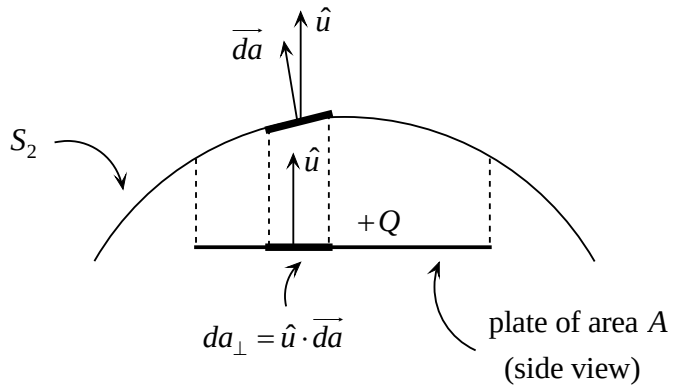

Figure 4. A side view of the positively-charged plate of the capacitor.

The dot product in the integral on the right of (A.8) represents the projection of the surface element $\overrightarrow{da}$ onto the axis defined by the unit vector $\hat{u}$ (see Fig. 4). This is equal to the projection $da_\perp$ of an elementary area da of S_2' onto the flat surface of the plate of the capacitor. Eventually, the integral on the right of (A.8) equals the total area A of the plate. Hence, $I_{d,in}=I(t)$ and, given that $I_{in}=0$ on S_2 , the Ampère-Maxwell law (A.3) again yields the result (A.7).

So, everything works fine with regard to the Ampère-Maxwell law, but there is one law we have not taken into account so far; namely, the *Faraday-Henry law*! According to that law, a time-changing magnetic field is always accompanied by an electric field (or, as is often said, "induces" an electric field). So, the electric field outside the capacitor cannot be zero, as claimed previously, given that the time-dependent current $I(t)$ is expected to generate a time-dependent magnetic field. For a similar reason, the electric field inside the capacitor cannot have the static-field-like form (A.4) (there must also be a contribution from the rate of change of the magnetic field between the plates).

An exception occurs if the current I that charges the capacitor is constant in time (i.e., if the capacitor is being charged at a constant rate) since in this case the magnetic field will be static everywhere. But, in the general case where $I(t)\neq$constant, the preceding discussion regarding the charging capacitor problem needs to be revised in order to take into account the entire set of Maxwell's equations; in particular, the Ampère-Maxwell law as well as the Faraday-Henry law.

Appendix III. General form of the electric field

To justify the general expression for the electric field implied in the *Ansatz* (5) used to find solutions of Maxwell's equations inside the capacitor, we need to prove the following:

Lemma 1. If the magnetic field inside the capacitor is azimuthal, of the form

$$\vec{B} = B(\rho, t)\hat{u}_\varphi \tag{A.9}$$

then the electric field (also assumed dependent on ρ and t) is of the form

$$\vec{E} = E(\rho, t)\hat{u}_z \tag{A.10}$$

Proof. Let

$$\vec{E} = E_\rho(\rho,t)\,\hat{u}_\rho + E_\varphi(\rho,t)\,\hat{u}_\varphi + E_z(\rho,t)\,\hat{u}_z \tag{A.11}$$

Then (cf. Appendix I) from Gauss' law (4*a*) it follows that

$$\frac{\partial}{\partial\rho}(\rho E_\rho) = 0 \;\Rightarrow\; E_\rho \equiv \frac{\alpha(t)}{\rho} \tag{A.12}$$

In order for the electric field to be finite at the center of the capacitor (i.e., for ρ=0) we must set $\alpha(t)\equiv 0$, so that $E_\rho(\rho,t)=0$. On the other hand, the *z*-component of Faraday's law (4*c*) yields

$$\frac{\partial}{\partial\rho}(\rho E_\varphi) = 0 \;\Rightarrow\; E_\varphi \equiv \frac{\beta(t)}{\rho} \tag{A.13}$$

Again, finiteness of the electric field for ρ=0 dictates that $\beta(t)\equiv 0$, so that $E_\varphi(\rho,t)=0$. Eventually, only the *z*-component of the electric field is non-vanishing, in accordance with (A.10).

The solutions outside the capacitor are subject to the restriction $\rho>0$. The expression for the electric field implied in the *Ansatz* (16) is based on the following observation:

Lemma 2. If the magnetic field outside the capacitor is azimuthal, of the form (A.9), then the electric field (also assumed dependent on ρ and *t*) is again of the form (A.10).

Proof. Let the electric field be of the form (A.11). Then from Gauss' law (4*a*) and from the *z*-component of Faraday's law (4*c*) we get (A.12) and (A.13), respectively. On the other hand, from the ρ- and φ-components of the fourth Maxwell equation (4*d*) we find that $\partial E_\rho/\partial t=0$ and $\partial E_\varphi/\partial t=0$, which means that α and β are actually constants. Thus the general form of the electric field outside the capacitor should be

$$\vec{E} = \frac{\alpha}{\rho}\hat{u}_\rho + \frac{\beta}{\rho}\hat{u}_\varphi + f(\rho,t)\,\hat{u}_z \;.$$

Obviously, the function $f(\rho,t)$ is related to the time-change of the magnetic field and is expected to vanish if the current *I* that charges the capacitor is constant. If the electric field itself is to vanish when *I=constant*, both constants α and β must be zero. Eventually, the electric field outside the capacitor must be of the general form (A.10).

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Mathematics > General Mathematics

Bäcklund Transformations: Some Old and New Perspectives

C. J. Papachristou, A. N. Magoulas

(Submitted on 20 Jul 2016)

B\"acklund transformations (BTs) are traditionally regarded as a tool for integrating nonlinear partial differential equations (PDEs). Their use has been recently extended, however, to problems such as the construction of recursion operators for symmetries of PDEs, as well as the solution of linear systems of PDEs. In this article, the concept and some applications of BTs are reviewed. As an example of an integrable linear system of PDEs, the Maxwell equations of electromagnetism are shown to constitute a BT connecting the wave equations for the electric and the magnetic field; plane-wave solutions of the Maxwell system are constructed in detail. The connection between BTs and recursion operators is also discussed.

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Bäcklund Transformations: Some Old and New Perspectives

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Abstract

Bäcklund transformations (BTs) are traditionally regarded as a tool for integrating nonlinear partial differential equations (PDEs). Their use has been recently extended, however, to problems such as the construction of recursion operators for symmetries of PDEs, as well as the solution of linear systems of PDEs. In this article, the concept and some applications of BTs are reviewed. As an example of an integrable linear system of PDEs, the Maxwell equations of electromagnetism are shown to constitute a BT connecting the wave equations for the electric and the magnetic field; plane-wave solutions of the Maxwell system are constructed in detail. The connection between BTs and recursion operators is also discussed.

1. Introduction

Bäcklund transformations (BTs) were originally devised as a tool for obtaining solutions of nonlinear partial differential equations (PDEs) (see, e.g., [1] and the references therein). They were later also proven useful as *recursion operators* for constructing infinite sequences of nonlocal symmetries and conservation laws of certain PDEs [2–6].

In simple terms, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs [say, (*a*) and (*b*)] in order for the system to be integrable for either field. If a solution of PDE (*a*) is known, then a solution of PDE (*b*) is obtained simply by integrating the BT, without having to actually solve the latter PDE (which, presumably, would be a much harder task). In the case where the PDEs (*a*) and (*b*) are identical, the *auto-BT* produces new solutions of PDE (*a*) from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. But, what if the BT itself is the differential system whose solutions we are looking for? As it turns out, to solve the problem we need to have parameter-dependent solutions of *both* PDEs (*a*) and (*b*) at hand. By properly matching the parameters (provided this is possible) a solution of the given system is obtained.

The above method is particularly effective in *linear* problems, given that parametric solutions of linear PDEs are generally not hard to find. An important paradigm of a BT associated with a linear problem is offered by the Maxwell system of equations of electromagnetism [7,8]. As is well known, the consistency of this system demands that both the electric and the magnetic field independently satisfy a respective wave equation. These equations have known, parameter-dependent solutions; namely, monochromatic plane waves with arbitrary amplitudes, frequencies and wave vectors

(the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the appropriate expressions for the "parameters" in order for the plane waves to also be solutions of Maxwell's equations; that is, in order to represent an actual electromagnetic field.

This article, written for educational purposes, is an introduction to the concept of a BT and its application to the solution of PDEs or systems of PDEs. Both "classical" and novel views of a BT are discussed, the former view predominantly concerning integration of nonlinear PDEs while the latter one being applicable mostly to linear systems of PDEs. The article is organized as follows:

In Section 2 we review the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 a different perception of a BT is presented, according to which it is the BT itself whose solutions are sought. The concept of *conjugate solutions* is introduced.

As an example, in Secs. 4 and 5 the Maxwell equations in empty space and in a linear conducting medium, respectively, are shown to constitute a BT connecting the wave equations for the electric and the magnetic field. Following [7], the process of constructing plane-wave solutions of this BT is presented in detail. This process is, of course, a familiar problem of electrodynamics but is seen here under a new perspective by employing the concept of a BT.

Finally, in Sec. 6 we briefly review the connection between BTs and recursion operators for generating infinite sequences of nonlocal symmetries of PDEs.

2. Bäcklund Transformations: Classical Viewpoint

Consider two PDEs $P[u]=0$ and $Q[v]=0$ for the unknown functions u and v, respectively. The expressions $P[u]$ and $Q[v]$ may contain the corresponding variables u and v, as well as partial derivatives of u and v with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, t. Partial derivatives with respect to these variables will be denoted by using subscripts: u_x, u_t, u_{xx}, u_{tt}, u_{xt}, etc.

Independently, for the moment, also consider a pair of coupled PDEs for u and v:

$$B_1[u,v]=0 \quad (a) \qquad B_2[u,v]=0 \quad (b) \tag{1}$$

where the expressions $B_i[u,v]$ $(i=1,2)$ may contain u, v as well as partial derivatives of u and v with respect to x and t. We note that u appears in both equations (*a*) and (*b*). The question then is: if we find an expression for u by integrating (*a*) for a given v, will it match the corresponding expression for u found by integrating (*b*) for the same v? The answer is that, in order that (*a*) and (*b*) be consistent with each other for solution for u, the function v must be properly chosen so as to satisfy a certain *consistency condition* (or *integrability condition* or *compatibility condition*).

By a similar reasoning, in order that (*a*) and (*b*) in (1) be mutually consistent for solution for v, for some given u, the function u must now itself satisfy a corresponding integrability condition.

If it happens that the two consistency conditions for integrability of the system (1) are precisely the PDEs $P[u]=0$ and $Q[v]=0$, we say that the above system constitutes a *Bäcklund transformation* (BT) connecting solutions of $P[u]=0$ with solutions of

$Q[v]=0$. In the special case where $P\equiv Q$, i.e., when u and v satisfy *the same* PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT) for this PDE.

Suppose now that we seek solutions of the PDE $P[u]=0$. Assume that we are able to find a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$, the auto-BT connects solutions u and v of the same PDE) and let $v=v_0(x,t)$ be some known solution of $Q[v]=0$. The BT is then a system of PDEs for the unknown u,

$$B_i[u, v_0] = 0 \ , \quad i = 1, 2 \tag{2}$$

The system (2) is integrable for u, given that the function v_0 satisfies *a priori* the required integrability condition $Q[v]=0$. The solution u then of the system satisfies the PDE $P[u]=0$. Thus a solution $u(x,t)$ of the latter PDE is found without actually solving the equation itself, simply by integrating the BT (2) with respect to u. Of course, this method will be useful provided that integrating the system (2) for u is simpler than integrating the PDE $P[u]=0$ itself. If the transformation (2) is an auto-BT for the PDE $P[u]=0$, then, starting with a known solution $v_0(x,t)$ of this equation and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of the use of a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of Complex Analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{3}$$

(here, the variable t has been renamed y) constitute an auto-BT for the *Laplace equation*,

$$P[w] \equiv w_{xx} + w_{yy} = 0 \tag{4}$$

Let us explain this: Suppose we want to solve the system (3) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (a) and (b) match for solution for u, we must compare them in some way. We thus differentiate (a) with respect to y and (b) with respect to x, and equate the mixed derivatives of u. That is, we apply the integrability condition $(u_x)_y = (u_y)_x$. In this way we eliminate the variable u and find the condition that must be obeyed by $v(x,y)$:

$$P[v] \equiv v_{xx} + v_{yy} = 0 \ .$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate v from the system (3), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$P[u] \equiv u_{xx} + u_{yy} = 0 \ .$$

In conclusion, the integrability of system (3) with respect to either variable requires that the other variable must satisfy the Laplace equation (4).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate this system with respect to u. It is not hard to

show (by eliminating v_0 from the system) that the solution u will also satisfy the Laplace equation (4). As an example, by choosing the solution $v_0(x,y)=xy$, we find a new solution $u(x,y)= (x^2 – y^2)/2 + C$.

2. The *Liouville equation* is written

$$P[u] \equiv u_{xt} - e^u = 0 \quad \Leftrightarrow \quad u_{xt} = e^u \tag{5}$$

Due to its nonlinearity, this PDE is hard to integrate directly. A solution is thus sought by means of a BT. We consider an auxiliary function $v(x,t)$ and an associated PDE,

$$Q[v] \equiv v_{xt} = 0 \tag{6}$$

We also consider the system of first-order PDEs,

$$u_x + v_x = \sqrt{2}\ e^{(u-v)/2} \quad (a) \qquad u_t - v_t = \sqrt{2}\ e^{(u+v)/2} \quad (b) \tag{7}$$

Differentiating the PDE (a) with respect to t and the PDE (b) with respect to x, and eliminating $(u_t – v_t)$ and $(u_x + v_x)$ in the ensuing equations with the aid of (a) and (b), we find that u and v satisfy the PDEs (5) and (6), respectively. Thus, the system (7) is a BT connecting solutions of (5) and (6). Starting with the trivial solution $v=0$ of (6), and integrating the system

$$u_x = \sqrt{2}\ e^{u/2} , \qquad u_t = \sqrt{2}\ e^{u/2} ,$$

we find a nontrivial solution of (5):

$$u(x,t) = -2\ln\left(C - \frac{x+t}{\sqrt{2}} \right) .$$

3. The *"sine-Gordon" equation* has applications in various areas of Physics, e.g., in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name is a pun on the related linear Klein-Gordon equation) is written

$$P[u] \equiv u_{xt} - \sin u = 0 \quad \Leftrightarrow \quad u_{xt} = \sin u \tag{8}$$

The following system of equations is an auto-BT for the nonlinear PDE (8):

$$\frac{1}{2}(u+v)_x = a\ \sin\left(\frac{u-v}{2} \right) , \qquad \frac{1}{2}(u-v)_t = \frac{1}{a}\ \sin\left(\frac{u+v}{2} \right) \tag{9}$$

where a ($\neq 0$) is an arbitrary real constant. [Because of the presence of a, the system (9) is called a *parametric* BT.] When u is a solution of (8) the BT (9) is integrable for v, which, in turn, also is a solution of (8): $P[v]=0$; and vice versa. Starting with the trivial solution $v=0$ of $v_{xt} = \sin v$, and integrating the system

$$u_x = 2a \sin\frac{u}{2} \quad , \quad u_t = \frac{2}{a} \sin\frac{u}{2} \quad ,$$

we obtain a new solution of (8):

$$u(x,t) = 4\arctan\left\{C\exp\left(ax + \frac{t}{a}\right)\right\} \ .$$

3. Conjugate Solutions and Another View of a BT

As presented in the previous section, a BT is an auxiliary device for constructing solutions of a (usually nonlinear) PDE from known solutions of the same or another PDE. The converse problem, where solutions of the differential system representing the BT itself are sought, is also of interest, however, and has been recently suggested [7,8] in connection with the Maxwell equations (see subsequent sections).

To be specific, assume that we need to integrate a given system of PDEs connecting two functions *u* and *v*:

$$B_i[u,v] = 0 \ , \quad i = 1,2 \tag{10}$$

Suppose that the integrability of the system for both functions requires that *u* and *v* separately satisfy the respective PDEs

$$P[u] = 0 \quad (a) \qquad Q[v] = 0 \quad (b) \tag{11}$$

That is, the system (10) is a BT connecting solutions of the PDEs (11). Assume, now, that these PDEs possess known (or, in any case, easy to find) *parameter-dependent solutions* of the form

$$u = f(x,y;\alpha,\beta,\ldots) \quad , \quad v = g(x,y;\kappa,\lambda,\ldots) \tag{12}$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which *u* and *v* jointly satisfy the system (10), we say that the solutions *u* and *v* of the PDEs (11*a*) and (11*b*), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short). By finding a pair of BT-conjugate solutions one thus automatically obtains a solution of the system (10).

Note that solutions of *both* integrability conditions $P[u]=0$ and $Q[v]=0$ must now be known in advance! From the practical point of view the method is thus most applicable in *linear* problems, since it is much easier to find parameter-dependent solutions of the PDEs (11) in this case.

Let us see an example: Going back to the Cauchy-Riemann relations (3), we try the following parametric solutions of the Laplace equation (4):

$$u(x,y) = \alpha(x^2 - y^2) + \beta x + \gamma y \ ,$$
$$v(x,y) = \kappa xy + \lambda x + \mu y \ .$$

Substituting these into the BT (3), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$u(x,y)=\alpha(x^2-y^2)+\beta x+\gamma y \ ,$$
$$v(x,y)=2\alpha xy-\gamma x+\beta y$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u(x,y)=\alpha xy \ , \quad v(x,y)=\beta xy \ .$$

Inserting these into the system (3) and taking into account the independence of x and y, we find that the only possible values of the parameters α and β are $\alpha=\beta=0$, so that $u(x,y)=v(x,y)=0$. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. Example: The Maxwell Equations in Empty Space

An example of an integrable linear system whose solutions are of physical interest is furnished by the *Maxwell equations* of electrodynamics. Interestingly, as noted recently [7], the Maxwell system has the property of a BT whose integrability conditions are the electromagnetic (e/m) wave equations that are separately valid for the electric and the magnetic field. These equations possess parameter-dependent solutions that, by a proper choice of the parameters, can be made BT-conjugate through the Maxwell system. In this and the following section we discuss the BT property of the Maxwell equations in vacuum and in a conducting medium, respectively.

In *empty space*, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units) [9]

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad && (c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad && (d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\,\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{13}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. Here we have a system of four PDEs for two fields. The question is: what are the necessary conditions that each of these fields must satisfy in order for the system (13) to be self-consistent? In other words, what are the *consistency conditions* (or *integrability conditions*) for this system?

Guided by our experience from Sec. 2, to find these conditions we perform various differentiations of the equations of system (13) and require that certain differential identities be satisfied. Our aim is, of course, to eliminate one field (electric or magnetic) in favor of the other and find some higher-order PDE that the latter field must obey.

As can be checked, two differential identities are satisfied automatically in the system (13):

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})=0\ ,\quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})=0\ ,$$

$$(\vec{\nabla}\cdot\vec{E})_t=\vec{\nabla}\cdot\vec{E}_t\ ,\quad (\vec{\nabla}\cdot\vec{B})_t=\vec{\nabla}\cdot\vec{B}_t\ .$$

Two others read

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{14}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{15}$$

Taking the *rot* of (13*c*) and using (14), (13*a*) and (13*d*), we find

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{16}$$

Similarly, taking the *rot* of (13*d*) and using (15), (13*b*) and (13*c*), we get

$$\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{17}$$

No new information is furnished by the remaining two integrability conditions,

$$(\vec{\nabla}\times\vec{E})_t=\vec{\nabla}\times\vec{E}_t\ ,\quad (\vec{\nabla}\times\vec{B})_t=\vec{\nabla}\times\vec{B}_t\ .$$

Note that we have *uncoupled* the equations for the two fields in the system (13), deriving separate second-order PDEs for each field. Putting

$$\varepsilon_0\mu_0\equiv\frac{1}{c^2}\ \Leftrightarrow\ c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \tag{18}$$

(where c is the speed of light in vacuum) we rewrite (16) and (17) in wave-equation form:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{19}$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{20}$$

We conclude that the Maxwell system (13) is a BT relating solutions of the e/m wave equations (19) and (20), these equations representing the integrability conditions of the BT. It should be noted that this BT is *not* an *auto*-BT! Indeed, although the PDEs (19) and (20) are of similar form, they concern *different* fields with different physical dimensions and physical properties.

The e/m wave equations admit plane-wave solutions of the form $\vec{F}(\vec{k}\cdot\vec{r}-\omega t)$, with

$$\frac{\omega}{k}=c \quad \text{where} \quad k=|\vec{k}| \tag{21}$$

The simplest such solutions are *monochromatic plane waves* of angular frequency ω, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \quad (a)\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \quad (b)\end{aligned} \tag{22}$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes. The constants appearing in the above equations (amplitudes, frequency and wave vector) can be chosen arbitrarily; thus they can be regarded as *parameters* on which the plane waves (22) depend.

We must note carefully that, although every pair of fields $(\vec{E},\vec{B})$ satisfying the Maxwell equations (13) also satisfies the wave equations (19) and (20), the converse is not true. Thus, the plane-wave solutions (22) are not *a priori* solutions of the Maxwell system (i.e., do not represent actual e/m fields). This problem can be taken care of, however, by a proper choice of the parameters in (22). To this end, we substitute the general solutions (22) into the BT (13) to find the extra conditions the latter system demands. By fixing the wave parameters, the two wave solutions in (22) will become *BT-conjugate* through the Maxwell system (13).

Substituting (22*a*) and (22*b*) into (13*a*) and (13*b*), respectively, and taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, we have

$$\begin{aligned}(\vec{E}_0\,e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;&\Rightarrow\; (\vec{k}\cdot\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\,,\\ (\vec{B}_0\,e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;&\Rightarrow\; (\vec{k}\cdot\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\,,\end{aligned}$$

so that

$$\vec{k}\cdot\vec{E}_0=0\,,\quad \vec{k}\cdot\vec{B}_0=0\,. \tag{23}$$

Relations (23) reflect the fact that that the monochromatic plane e/m wave is a *transverse wave*.

Next, substituting (22*a*) and (22*b*) into (13*c*) and (13*d*), we find

$$\begin{aligned}&e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{E}_0=i\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\;\Rightarrow\\ &(\vec{k}\times\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,\end{aligned}$$

$$\begin{aligned}&e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{B}_0=-i\omega\,\varepsilon_0\mu_0\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\;\Rightarrow\\ &(\vec{k}\times\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=-\frac{\omega}{c^2}\,\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,\end{aligned}$$

so that

$$\vec{k}\times\vec{E}_0=\omega\,\vec{B}_0\ ,\quad \vec{k}\times\vec{B}_0=-\frac{\omega}{c^2}\,\vec{E}_0 \tag{24}$$

We note that the fields $\vec{E}$ and $\vec{B}$ are normal to each other, as well as normal to the direction of propagation of the wave. We also remark that the two vector equations in (24) are not independent of each other, since, by cross-multiplying the first relation by $\vec{k}$, we get the second relation.

Introducing a unit vector $\hat{\tau}$ in the direction of the wave vector $\vec{k}$,

$$\hat{\tau}=\vec{k}/k\quad (k=|\vec{k}|=\omega/c)\ ,$$

we rewrite the first of equations (24) as

$$\vec{B}_0=\frac{k}{\omega}(\hat{\tau}\times\vec{E}_0)=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\ .$$

The BT-conjugate solutions in (22) are now written

$$\begin{aligned}&\vec{E}(\vec{r},t)=\vec{E}_0\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}\ ,\\ &\vec{B}(\vec{r},t)=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{25}$$

As constructed, the complex vector fields in (25) satisfy the Maxwell system (13). Since this system is homogeneous linear with real coefficients, the real parts of the fields (25) also satisfy it. To find the expressions for the real solutions (which, after all, carry the physics of the situation) we take the simplest case of *linear polarization* and write

$$\vec{E}_0=\vec{E}_{0,R}\,e^{\,i\alpha} \tag{26}$$

where the vector $\vec{E}_{0,R}$ as well as the number α are real. The *real* versions of the fields (25), then, read

$$\begin{aligned}&\vec{E}=\vec{E}_{0,R}\cos(\vec{k}\cdot\vec{r}\,-\omega\,t+\alpha)\ ,\\ &\vec{B}=\frac{1}{c}(\hat{\tau}\times\vec{E}_{0,R})\cos(\vec{k}\cdot\vec{r}\,-\omega\,t+\alpha)=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{27}$$

We note, in particular, that the fields $\vec{E}$ and $\vec{B}$ "oscillate" in phase.

Our results for the Maxwell equations in vacuum can be extended to the case of a *linear non-conducting medium* upon replacement of ε_0 and μ_0 with ε and μ, respectively. The speed of propagation of the e/m wave is, in this case,

$$\upsilon=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon\mu}}\ .$$

In the next section we study the more complex case of a linear medium having a finite conductivity.

5. Example: The Maxwell System for a Linear Conducting Medium

Consider a linear conducting medium of conductivity σ. In such a medium, Ohm's law is satisfied: $\vec{J}_f = \sigma \vec{E}$, where $\vec{J}_f$ is the free current density. The Maxwell equations take on the form [9]

$$\begin{aligned} &(a) \quad \vec{\nabla} \cdot \vec{E} = 0 \qquad && (c) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &(b) \quad \vec{\nabla} \cdot \vec{B} = 0 \qquad && (d) \quad \vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \varepsilon \mu \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{28}$$

By requiring satisfaction of the integrability conditions

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \,, \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} \,, \end{aligned}$$

we obtain the *modified wave equations*

$$\begin{aligned} \nabla^2 \vec{E} - \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \sigma \frac{\partial \vec{E}}{\partial t} = 0 \\ \nabla^2 \vec{B} - \varepsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} - \mu \sigma \frac{\partial \vec{B}}{\partial t} = 0 \end{aligned} \tag{29}$$

which must be separately satisfied by each field. As in Sec. 4, no further information is furnished by the remaining integrability conditions.

The linear differential system (28) is a BT relating solutions of the wave equations (29). As in the vacuum case, this BT is *not* an auto-BT. We now seek BT-conjugate solutions. As can be verified by direct substitution into equations (29), these PDEs admit parameter-dependent solutions of the form

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \exp\{-s\,\hat{\tau} \cdot \vec{r} + i\,(\vec{k} \cdot \vec{r} - \omega t)\} \\ &= \vec{E}_0 \exp\left\{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}\right\} \exp(-i\omega t) \,, \\ \vec{B}(\vec{r},t) &= \vec{B}_0 \exp\{-s\,\hat{\tau} \cdot \vec{r} + i\,(\vec{k} \cdot \vec{r} - \omega t)\} \\ &= \vec{B}_0 \exp\left\{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}\right\} \exp(-i\omega t) \end{aligned} \tag{30}$$

where $\hat{\tau}$ is the unit vector in the direction of the wave vector $\vec{k}$:

$$\hat{\tau} = \vec{k}/k \quad (k = |\vec{k}| = \omega/\upsilon)$$

(υ is the speed of propagation of the wave inside the conducting medium) and where, for given physical characteristics ε, μ, σ of the medium, the parameters s, k and ω satisfy the algebraic system

$$s^2 - k^2 + \varepsilon\mu\omega^2 = 0 \ , \quad \mu\sigma\omega - 2sk = 0 \tag{31}$$

We note that, for arbitrary choices of the amplitudes $\vec{E}_0$ and $\vec{B}_0$, the vector fields (30) are not *a priori* solutions of the Maxwell system (28), thus are not BT-conjugate solutions. To obtain such solutions we substitute expressions (30) into the system (28). With the aid of the relation

$$\vec{\nabla} e^{\left(i - \frac{s}{k}\right)\vec{k}\cdot\vec{r}} = \left(i - \frac{s}{k}\right)\vec{k}\, e^{\left(i - \frac{s}{k}\right)\vec{k}\cdot\vec{r}}$$

one can show that (28*a*) and (28*b*) impose the conditions

$$\vec{k}\cdot\vec{E}_0 = 0 \ , \quad \vec{k}\cdot\vec{B}_0 = 0 \tag{32}$$

As in the vacuum case, the e/m wave in a conducting medium is a *transverse* wave.

By substituting (30) into (28*c*) and (28*d*), two more conditions are found:

$$(k + is)\,\hat{\tau}\times\vec{E}_0 = \omega\vec{B}_0 \tag{33}$$

$$(k + is)\,\hat{\tau}\times\vec{B}_0 = -(\varepsilon\mu\omega + i\mu\sigma)\vec{E}_0 \tag{34}$$

Note, however, that (34) is not an independent equation since it can be reproduced by cross-multiplying (33) by $\hat{\tau}$, taking into account the algebraic relations (31).

The BT-conjugate solutions of the wave equations (29) are now written

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0\, e^{-s\hat{\tau}\cdot\vec{r}} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \ , \\ \vec{B}(\vec{r},t) &= \frac{k+is}{\omega}\,(\hat{\tau}\times\vec{E}_0)\, e^{-s\hat{\tau}\cdot\vec{r}} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{aligned} \tag{35}$$

To find the corresponding real solutions, we assume linear polarization of the wave, as before, and set

$$\vec{E}_0 = \vec{E}_{0,R}\, e^{i\alpha} \ .$$

We also put

$$k + i\,s = |k + i\,s|\, e^{i\varphi} = \sqrt{k^2 + s^2}\ e^{i\varphi} \ ; \ \tan\varphi = s/k \ .$$

Taking the real parts of equations (35), we finally have:

$$\vec{E}(\vec{r},t) = \vec{E}_{0,R}\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha)\ ,$$

$$\vec{B}(\vec{r},t) = \frac{\sqrt{k^2+s^2}}{\omega}\, (\hat{\tau}\times\vec{E}_{0,R})\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha + \varphi)\ .$$

As an exercise, the student may show that these results reduce to those for a linear non-conducting medium (cf. Sec. 4) in the limit $\sigma\to 0$.

6. BTs as Recursion Operators

The concept of symmetries of PDEs was discussed in [1]. Let us review the main facts:

Consider a PDE $F[u]=0$, where, for simplicity, $u=u(x,t)$. A transformation

$$u(x,t) \;\to\; u'(x,t)$$

from the function u to a new function u' represents a *symmetry* of the given PDE if the following condition is satisfied: $u'(x,t)$ is a solution of $F[u]=0$ *if* $u(x,t)$ is a solution. That is,

$$F[u'] = 0 \quad \textit{when} \quad F[u] = 0 \tag{36}$$

An *infinitesimal symmetry transformation* is written

$$u' = u + \delta u = u + \alpha Q[u] \tag{37}$$

where α is an infinitesimal parameter. The function $Q[u]\equiv Q(x, t, u, u_x, u_t, ...)$ is called the *symmetry characteristic* of the transformation (37).

In order that a function $Q[u]$ be a symmetry characteristic for the PDE $F[u]=0$, it must satisfy a certain PDE that expresses the *symmetry condition* for $F[u]=0$. We write, symbolically,

$$S(Q;u) = 0 \quad \textit{when} \quad F[u] = 0 \tag{38}$$

where the expression S depends *linearly* on Q and its partial derivatives. Thus, (38) is a linear PDE for Q, in which equation the variable u enters as a sort of parametric function that is required to satisfy the PDE $F[u]=0$.

A *recursion operator* $\hat{R}$ [10] is a linear operator which, acting on a symmetry characteristic Q, produces a new symmetry characteristic $Q' = \hat{R}Q$. That is,

$$S(\hat{R}Q;u) = 0 \quad \textit{when} \quad S(Q;u) = 0 \tag{39}$$

It is not too difficult to show that *any power of a recursion operator also is a recursion operator*. This means that, starting with any symmetry characteristic Q, one may

in principle obtain an infinite set of characteristics (thus, an infinite number of symmetries) by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [2,3] (see also [4-6]). According to this view, a recursion operator is an auto-BT for the linear PDE (38) expressing the symmetry condition of the problem; that is, a BT producing new solutions Q' of (38) from old ones, Q. Typically, this type of BT produces *nonlocal* symmetries, i.e., symmetry characteristics depending on *integrals* (rather than derivatives) of u.

As an example, consider the *chiral field equation*

$$F[g] \equiv (g^{-1}g_x)_x + (g^{-1}g_t)_t = 0 \tag{40}$$

(as usual, subscripts denote partial differentiations) where g is a $GL(n,C)$-valued function of x and t (i.e., an invertible complex $n\times n$ matrix, differentiable for all x, t).

Let $Q[g]$ be a symmetry characteristic of the PDE (40). It is convenient to put

$$Q[g] = g\,\Phi[g]$$

and write the corresponding infinitesimal symmetry transformation in the form

$$g' = g + \delta g = g + \alpha g\,\Phi[g] \tag{41}$$

The symmetry condition that Q must satisfy will be a PDE linear in Q, thus in Φ also. As can be shown [4], this PDE is

$$S(\Phi; g) \equiv \Phi_{xx} + \Phi_{tt} + [g^{-1}g_x\,,\,\Phi_x] + [g^{-1}g_t\,,\,\Phi_t] = 0 \tag{42}$$

which must be valid when $F[g]=0$ (where, in general, $[A, B] \equiv AB–BA$ denotes the *commutator* of two matrices A and B).

For a given g satisfying $F[g]=0$, consider now the following system of PDEs for the matrix functions Φ and Φ':

$$\begin{aligned} \Phi'_x &= \Phi_t + [g^{-1}g_t\,,\,\Phi] \\ -\Phi'_t &= \Phi_x + [g^{-1}g_x\,,\,\Phi] \end{aligned} \tag{43}$$

The integrability condition $(\Phi'_x)_t = (\Phi'_t)_x$, together with the equation $F[g]=0$, require that Φ be a solution of (42): $S(\Phi\,;g) = 0$. Similarly, by the integrability condition $(\Phi_t)_x = (\Phi_x)_t$ one finds, after a lengthy calculation: $S(\Phi';g) = 0$.

In conclusion, for any g satisfying the PDE (40), the system (43) is a BT relating solutions Φ and Φ' of the symmetry condition (42) of this PDE; that is, relating different symmetries of the chiral field equation (40). Thus, if a symmetry characteristic $Q=g\Phi$ of (40) is known, a new characteristic $Q'=g\Phi'$ may be found by integrating the BT (43); the converse is also true. Since the BT (43) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (40).

As an example, for any constant matrix M the choice $\Phi=M$ clearly satisfies the symmetry condition (42). This corresponds to the symmetry characteristic $Q=gM$. By integrating the BT (43) for Φ', we get $\Phi'=[X, M]$ and $Q'=g[X, M]$, where X is the "potential" of the PDE (40), defined by the system of PDEs

$$X_x = g^{-1} g_t \ , \quad -X_t = g^{-1} g_x \tag{44}$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (44), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (43), leading to an infinite sequence of increasingly nonlocal symmetries.

7. Summary

Classically, Bäcklund transformations (BTs) have been developed as a useful tool for finding solutions of nonlinear PDEs, given that these equations are usually hard to solve by direct methods. By means of examples we saw that, starting with even the most trivial solution of a PDE, one may produce a highly nontrivial solution of this (or another) PDE by integrating the BT, without solving the original, nonlinear PDE directly (which, in most cases, is a much harder task).

A different use of BTs, that was recently proposed [7,8], concerns predominantly the solution of linear systems of PDEs. This method relies on the existence of parameter-dependent solutions of the linear PDEs expressing the integrability conditions of the BT. This time it is the BT itself (rather than its associated integrability conditions) whose solutions are sought.

An appropriate example for demonstrating this approach to the concept of a BT is furnished by the Maxwell equations of electromagnetism. We showed that this system of PDEs can be treated as a BT whose integrability conditions are the wave equations for the electric and the magnetic field. These wave equations have known, parameter-dependent solutions – monochromatic plane waves – with arbitrary amplitudes, frequencies and wave vectors playing the roles of the "parameters". By substituting these solutions into the BT, one may determine the required relations among the parameters in order that these plane waves also represent electromagnetic fields (i.e., in order that they be solutions of the Maxwell system). The results arrived at by this method are, of course, well known in advanced electrodynamics. The process of deriving them, however, is seen here in a new light by employing the concept of a BT.

BTs have also proven useful as *recursion operators* for deriving infinite sets of nonlocal symmetries and conservation laws of PDEs [2-6] (see also [11] and the references therein). Specifically, the BT produces an increasingly nonlocal sequence of symmetry characteristics, i.e., solutions of the linear equation expressing the symmetry condition (or "linearization") of a given PDE.

An interesting conclusion is that the concept of a BT, which has been proven useful for integrating nonlinear PDEs, may also have important applications in linear problems. Research on these matters is in progress.

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Independence of Maxwell's equations: A Bäcklund-transformation view

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Abstract. It is now widely accepted that the Maxwell equations of Electrodynamics constitute a self-consistent set of four independent partial differential equations. According to a certain school of thought, however, half of these equations – namely, those expressing the two Gauss' laws for the electric and the magnetic field – are redundant since they can be "derived" from the remaining two laws and the principle of conservation of charge. The status of the latter principle is thus elevated to a law of Nature more fundamental than, say, Coulomb's law. In this note we examine this line of reasoning and we propose an approach according to which the Maxwell equations may be viewed as a Bäcklund transformation relating fields and sources. The conservation of charge and the electromagnetic wave equations then simply express the integrability conditions of this transformation.

Keywords: Classical electrodynamics, Maxwell's equations, Bäcklund transformations

1. Is Gauss' law of Electrodynamics redundant?

As we know, the *Maxwell equations* describe the behavior (that is, the laws of change in space and time) of the electromagnetic (e/m) field. This field is represented by the pair $(\vec{E},\vec{B})$, where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. The Maxwell equations additionally impose certain boundary conditions at the interface of two different media, while certain other physical demands are obvious (for example, the e/m field must vanish away from its localized "sources", unless these sources emit e/m radiation).

The Maxwell equations are a system of four partial differential equations (PDEs) that is self-consistent, in the sense that these equations are compatible with one another. The self-consistency of the system also implies the satisfaction of two important conditions that are physically meaningful:

- the *equation of continuity*, related to *conservation of charge*; and
- the *e/m wave equation* in its various forms.

We stress that these conditions are *necessary but not sufficient* for the validity of the Maxwell system. Thus, although every solution $(\vec{E},\vec{B})$ of this system obeys a wave equation separately for the electric and the magnetic field, an arbitrary pair of fields

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$(\vec{E},\vec{B})$, each field satisfying the corresponding wave equation, does not necessarily satisfy the Maxwell system itself. Also, the principle of conservation of charge *cannot replace* any one of Maxwell's equations. These remarks are justified by the fact that the aforementioned two necessary conditions are derived by *differentiating* the Maxwell system and, in this process, part of the information carried by this system is lost. [Recall, similarly, that cross-differentiation of the Cauchy-Riemann relations of complex analysis yields the Laplace equation (see Sec. 2) by which, however, we cannot recover the Cauchy-Riemann relations.]

The differential form of the Maxwell equations is

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad && (c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 && (d)\quad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where $\rho, \vec{J}$ are the charge and current densities, respectively (the "sources" of the e/m field). Both the fields and the sources are functions of the spacetime variables (*x*,*y*,*z*,*t*). Equations (1*a*) and (1*b*), which describe the *div* of the e/m field at any moment, constitute *Gauss' law* for the electric and the magnetic field, respectively. In terms of physical content, (1*a*) expresses the Coulomb law of electricity, while (1*b*) rules out the possibility of existence of magnetic poles analogous to electric charges. Equation (1*c*) expresses the *Faraday-Henry law* (law of e/m induction) and Eq. (1*d*) expresses the *Ampère-Maxwell law*. Equations (1*a*) and (1*d*), which contain the sources of the e/m field, constitute the *non-homogeneous* Maxwell equations, while Eqs. (1*b*) and (1*c*) are the *homogeneous* equations of the system.

By taking the *div* of (1*d*) and by using (1*a*), we obtain the *equation of continuity*, which physically expresses the *principle of conservation of charge* (see, e.g., [1], Sec. 9.6):

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \tag{2}$$

Although the charge and current densities on the right-hand sides of (1*a*) and (1*d*) are chosen freely and are considered known from the outset, relation (2) places a severe restriction on the associated functions. A different kind of differentiation of the Maxwell system (1), by taking the *rot* of (*c*) and (*d*), leads to separate wave equations (or modified wave equations, depending on the medium) for the electric and the magnetic field (see, e.g., [1], Sec. 10.4).

In most textbooks on electromagnetism (e.g., [2–6] and many more) the Maxwell equations (1) are treated as a consistent set of four independent PDEs. A number of authors, however, have doubted the independence of this system. Specifically, they argue that (1*a*) and (1*b*) – the equations for the *div* of the e/m field, expressing Gauss' law for the corresponding fields – are redundant since they "may be derived" from (1*c*) and (1*d*) in combination with the equation of continuity (2). If this is true, Coulomb's law – the most important experimental law of electricity – loses its status as an independent law and is reduced to a derivative theorem. The same can be said with regard to the non-existence of magnetic poles in Nature.

As far as we know, the first who doubted the independent status of the two Gauss' laws in electrodynamics was Julius Adams Stratton in his 1941 famous (and, admittedly, very attractive) book [7]. His reasoning may be described as follows:

By taking the *div* of (1*c*), the left-hand side vanishes identically while on the right-hand side we may change the order of differentiation with respect to space and time variables. The result is:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right)=0 \tag{3}$$

On the other hand, by taking the *div* of (1*d*) and by using the equation of continuity (2), we find that

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E}-\frac{\rho}{\varepsilon_0}\right)=0 \tag{4}$$

And the line of argument continues as follows: According to (3) and (4), the quantities $\vec{\nabla}\cdot\vec{B}$ and $(\vec{\nabla}\cdot\vec{E}-\rho/\varepsilon_0)$ are constant in time at every point (x,y,z) of the region Ω of space that concerns us. *If* we now assume that there has been a period of time during which no e/m field existed in the region Ω, then, in that period,

$$\vec{\nabla}\cdot\vec{B}=0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{E}-\rho/\varepsilon_0=0 \tag{5}$$

identically. Later on, although an e/m field did appear in Ω, the left-hand sides in (5) continued to vanish everywhere within this region since, as we said above, those quantities are time constant at every point of Ω. Thus, by the equations for the *rot* of the e/m field and by the principle of conservation of charge – the status of which was elevated from derivative theorem to fundamental law of the theory – we derived Eqs. (5), which are precisely the first two Maxwell equations (1*a*) and (1*b*)!

According to this reasoning, the electromagnetic theory is not based on four independent Maxwell equations but rather on *three* independent equations only; namely, the Faraday-Henry law (1*c*), the Ampère–Maxwell law (1*d*), and the principle of conservation of charge (2).

What makes this view questionable is the assumption that, for *every* region Ω of space there exists some period of time during which the e/m field in Ω vanishes. This hypothesis is arbitrary and is not dictated by the theory itself. (It is likely that no such region exists in the Universe!) Therefore, the argument that led from relations (3) and (4) to relations (5) is not convincing since it was based on an arbitrary and, in a sense, artificial initial condition: that the e/m field is zero at some time $t=0$ and before.

Let us assume for the sake of argument, however, that there exists a region Ω within which the e/m field is zero for $t<t_0$ and nonzero for $t>t_0$. The critical issue is what happens at $t=t_0$; specifically, whether the functions expressing the e/m field are *continuous* at that moment. If they indeed are, the field starts from zero and gradually increases to nonzero values; thus, the line of reasoning that led from (3) and (4) to (5) is acceptable. There are physical situations, however, in which the appearance of an e/m field is so abrupt that it may be considered *instantaneous*. (For instance, the moment we connect the ends of a metal wire to a battery, an electric field suddenly appears in the interior of the wire and a magnetic field appears in the exterior. An

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even more "dramatic" example is *pair production* in which a charged particle and the corresponding antiparticle are created simultaneously, thus an e/m field appears at that moment in the region.) In such cases the e/m field is *non-continuous* at $t=t_0$ and its time derivative is *not defined* at this instant. Therefore, the line of reasoning that leads from (3) and (4) to (5) again collapses.

Note, finally, a circular reasoning in Stratton's approach. It is assumed that, in a region Ω where no e/m field exists, the second of relations (5) is valid identically. This means that the vanishing of the electric field in Ω automatically implies the absence of electric charge in that region. This fact, however, follows from Gauss' law (1*a*); thus it may not be used *a priori* as a tool for proving the law itself!

Regarding charge conservation, we mentioned earlier that Eq. (2) is derived from the two non-homogeneous Maxwell equations, namely, Gauss' law (1*a*) for the electric field, and the Ampère–Maxwell law (1*d*). This means that the principle of conservation of charge is a *necessary* condition in order for the Maxwell system to be self-consistent. This condition is *not sufficient*, however, in the sense that it cannot replace any one of the system equations. Indeed, by the Ampère–Maxwell law and the conservation of charge there follows the *time derivative* of Gauss' law for the electric field [Eq. (4)]; this, however, does not imply that Gauss' law itself is valid. Of course, the reverse is true: *because* Gauss' law is valid, the same is true for its time derivative.

Our view, therefore, is that the Maxwell equations form a system of four independent PDEs that express respective laws of Nature. Moreover, the self-consistency of this system imposes two *necessary* (but *not sufficient*) conditions that concern the conservation of charge and the wave behavior of the time-dependent e/m field. In the next section the problem is re-examined from the point of view of Bäcklund transformations.

2. A Bäcklund-transformation view of Maxwell's equations

In previous articles [8,9] we suggested that, mathematically speaking, the Maxwell equations in empty space may be viewed as a Bäcklund transformation (BT) relating the electric and the magnetic field to each other. Let us briefly summarize a few key points regarding this idea. To begin with, let us see the simplest, perhaps, example of a BT.

The *Cauchy-Riemann relations* of complex analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{6}$$

(where subscripts denote partial derivatives with respect to the indicated variables) constitute a BT for the *Laplace equation*,

$$w_{xx} + w_{yy} = 0 \tag{7}$$

Let us explain this: Suppose we want to solve the system (6) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (6*a*) and (6*b*) match for solution for u, we must compare them in some way. We thus differentiate (6*a*) with respect to y and (6*b*) with respect to x, and equate the mixed derivatives of u. That is, we apply the *integrability condition* (or *consistency condition*) $(u_x)_y = (u_y)_x$. In this way we eliminate the variable u and we find a condition that must be obeyed by $v(x,y)$:

$$v_{xx} + v_{yy} = 0.$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate v from the system (6), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$u_{xx} + u_{yy} = 0.$$

In conclusion, the integrability of system (6) with respect to either variable requires that the other variable satisfy the Laplace equation (7).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (7). Substituting $v=v_0$ in the system (6), we can integrate this system with respect to u. It is not hard to show (by eliminating v_0 from the system) that the solution u will also satisfy the Laplace equation. As an example, by choosing the solution $v_0(x,y)=xy$ of (7), we find a new solution $u(x,y)=(x^2-y^2)/2+C$.

Generally speaking, a BT is a system of PDEs connecting two functions (say, u and v) in such a way that the consistency of the system requires that u and v independently satisfy the respective, higher-order PDEs $F[u]=0$ and $G[v]=0$. Analytically, in order that the system be integrable for u, the function v must be a solution of $G[v]=0$; conversely, in order that the system be integrable for v, the function u must be a solution of $F[u]=0$. If F and G happen to be functionally identical, as in the example given above, the BT is said to be an *auto-Bäcklund* transformation (auto-BT).

Classically, BTs are useful tools for finding solutions of nonlinear PDEs. In [8,9], however, we suggested that BTs may also be useful for solving *linear systems* of PDEs. The prototype example that we used was the Maxwell equations in empty space:

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &&(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &&(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{8}$$

Here we have a system of four PDEs for two vector fields that are functions of the spacetime coordinates (x,y,z,t). We would like to find the integrability conditions necessary for self-consistency of the system (8). To this end, we try to uncouple the system to find separate second-order PDEs for $\vec{E}$ and $\vec{B}$, the PDE for each field being a necessary condition in order that the system (8) be integrable for the other field. This uncoupling, which eliminates either field (electric or magnetic) in favor of the other, is achieved by properly differentiating the system equations and by using suitable vector identities, in a manner similar in spirit to that which took us from the first-order Cauchy-Riemann system (6) to the separate second-order Laplace equations (7) for u and v.

As discussed in [8,9], the only nontrivial integrability conditions for the system (8) are those obtained by using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{9}$$

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$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} \tag{10}$$

By these we obtain separate wave equations for the electric and the magnetic field:

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0 \tag{11}$$

$$\nabla^2\vec{B} - \varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2} = 0 \tag{12}$$

We conclude that the Maxwell system (8) in empty space is a BT relating the e/m wave equations for the electric and the magnetic field, in the sense that the wave equation for each field is an integrability condition for solution of the system in terms of the other field.

The case of the full Maxwell equations (1) is more complex due to the presence of the source terms $\rho, \vec{J}$ in the non-homogeneous equations (1a) and (1d). As it turns out, the self-consistency of the BT imposes restrictions on the terms of non-homogeneity as well as on the fields themselves. Before we get to this, however, let us see a simpler "toy" example that generalizes that of the Cauchy-Riemann relations.

Consider the following non-homogeneous linear system of PDEs for the functions u and v of the variables x, y, z, t:

$$\begin{aligned} u_x &= v_y \quad (a) & u_z &= v_z + p(x,y,z,t) \quad (c) \\ u_y &= -v_x \quad (b) & u_t &= v_t + q(x,y,z,t) \quad (d) \end{aligned} \tag{13}$$

where p and q are assumed to be given functions. The necessary consistency conditions for this system are found by cross-differentiation of the system equations with respect to the variables x, y, z, t. In particular, by cross-differentiating (a) and (b) with respect to x and y we find that $u_{xx}+u_{yy}=0$ and $v_{xx}+v_{yy}=0$; hence both u and v must satisfy the Laplace equation (7). On the other hand, cross-differentiation of (c) and (d) with respect to z and t eliminates the fundamental variables u and v, yielding a necessary condition for the terms of non-homogeneity, p and q; that is, $p_t - q_z = 0$. This means that the functions p and q cannot be chosen arbitrarily from the outset but must conform to this latter condition in order for the system (13) to have a solution.

As an application, let us take $v=xy+zt$ (which satisfies the Laplace equation $v_{xx}+v_{yy}=0$) and let us choose $p=2t$ and $q=2z$ (so that $p_t - q_z = 0$). It is not hard to show that the solution of the system (13) for u is then given by

$$u(x,y,z,t) = (x^2 - y^2)/2 + 3zt + C\,.$$

Notice that $u_{xx}+u_{yy}=0$, as expected.

Let us now return to the full Maxwell equations (1), which we now view as a BT relating the electric and the magnetic field and containing additional terms in which only the sources appear. As can be checked, there are now three nontrivial integrability conditions, namely, those found by applying the vector identities (9) and (10), as well as the identity

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)=0 \tag{14}$$

(the corresponding one for $\vec{E}$ is trivially satisfied in view of the Maxwell system). By (9) and (10) we get the non-homogeneous wave equations

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=\frac{1}{\varepsilon_0}\vec{\nabla}\rho+\mu_0\frac{\partial\vec{J}}{\partial t} \tag{15}$$

$$\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=-\mu_0\vec{\nabla}\times\vec{J} \tag{16}$$

Additionally, the integrability condition (14) yields the equation of continuity (2),

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \tag{17}$$

expressing conservation of charge. Notice that, unlike (15) and (16), the condition (17) places *a priori* restrictions *on the sources* rather than on the fields themselves!

In any case, the three relations (15) – (17) are *necessary* conditions imposed by the requirement of self-consistency of the BT (1). As explained in Sec. 1, however, these conditions are *not sufficient*, in the sense that none of them may replace any equation in the system (1). In particular, the equation of continuity (17) may not be regarded as more fundamental than the Gauss law (1*a*) for the electric field.

3. Conclusions

Let us summarize our main conclusions:

1. The Maxwell equations (1) express four separate laws of Nature. These equations are mathematically consistent with one another but constitute a set of independent vector relations, in the sense that no single equation may be deduced by the remaining three. In particular, the physical arguments that attempt to render the two Gauss' laws "redundant" are seen to be artificial and unrealistic.

2. We consider the Maxwell equations as physically acceptable simply because the system (1) and all conclusions mathematically drawn from it represent experimentally verifiable situations in Nature. Among these conclusions are the conservation of charge and the conservation of energy (Poynting's theorem). It should be kept in mind, however, that conservation laws appear as *consequences* of the fundamental equations of a theory, and not vice versa. In particular, conservation of charge, in the form of the continuity equation (17), is a physically verifiable mathematical conclusion drawn from the Maxwell system (1) but it may not be regarded as more fundamental than any equation in the system. The same can be said with regard to the existence of e/m waves, expressed mathematically by Eqs. (11) and (12).

3. From a mathematical perspective, the Maxwell system (1) may be viewed as a Bäcklund transformation (BT) the integrability conditions of which (i.e., the *necessary* conditions for self-consistency of the system) yield separate (generally non-

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homogeneous) wave equations (15) and (16) for the electric and the magnetic field, respectively, as well as the equation of continuity (17). These integrability conditions are derived by *differentiating* the BT in different ways; hence they carry less information than the BT itself. Consequently, none of the integrability conditions may replace any equation in the Maxwell system.

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On Non-Monochromatic Plane-Wave Solutions of Maxwell's Equations

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Abstract. In electrodynamics courses and textbooks the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this article the more general, non-monochromatic case is examined, which better exploits the independence of the Maxwell equations and offers new insights in the study of electromagnetic waves.

Keywords: Maxwell equations, electromagnetic waves, plane waves

Introduction

Plane electromagnetic (e/m) waves constitute a significant type of solution of the time-dependent Maxwell equations. A standard textbook approach, both at the intermediate level (e.g., Griffiths, 2013; Wangsness, 1986; Papachristou, 2020) and the advanced level (e.g., Jackson, 1999; Greiner, 1998), is to study the properties of the prototype monochromatic plane wave in a conducting or a non-conducting medium. This article examines the more general and much more interesting case of non-monochromatic plane e/m waves.

Relatively recently the Maxwell system of equations was shown to be a sort of *Bäcklund transformation* (BT for short; see Appendix) relating the e/m wave equations for the electric and the magnetic field (Papachristou, 2015 and 2019; Papachristou and Magoulas, 2016). Considered as a BT, the Maxwell system is a set of first-order partial differential equations (PDEs) the self-consistency of which set requires that the electric and the magnetic field separately satisfy a certain higher-order PDE, namely, a wave equation whose form depends on the medium of propagation of the e/m wave. Technically speaking, the wave equations for the electric and the magnetic field are *integrability conditions* (or *consistency conditions*) in order for the BT to be integrable for either field when the other field is known. We say that the solutions of the wave equations for the electric and the magnetic field, which jointly satisfy the Maxwell system, are *conjugate* through the BT.

A method for finding pairs $(\vec{E}, \vec{B})$ of conjugate solutions is the following: Since the wave equation is linear, it is not hard to find general, parameter-dependent solutions for the *E*-field and the *B*-field. In order that these solutions together satisfy the BT, the parameters of the two fields must be appropriately related. To establish this relation we substitute the parametric solutions into the BT (that is, into the Maxwell system) and find the necessary conditions in order that the *E*-field parameters and the *B*-field parameters match. The parametric pair $(\vec{E}, \vec{B})$, then, satisfies the Maxwell system and represents an e/m wave.

The simplest application of this method concerns the well-known case of a monochromatic plane wave of angular frequency ω (see, e.g., Papachristou and Magoulas, 2016). The frequency ω together with the wave vector and the amplitudes

of the *E* and *B*-fields constitute the "parameters" of the problem, which must be properly related in order that the pair $(\vec{E}, \vec{B})$ satisfy the Maxwell system.

The choice of simple monochromatic solutions, however, does not fully exploit the independence of the Maxwell equations (Papachristou and Magoulas, 2022). In this article the more general, non-monochromatic case is studied that makes minimal initial assumptions regarding the specific functional forms of the plane waves representing the electric and the magnetic field. The only assumption one does need to make from the outset is that both fields (electric and magnetic) are expressible in Fourier-integral form as linear superpositions of monochromatic waves of various frequencies. In particular, it is not even necessary to *a priori* require that the plane waves representing the two fields travel in the same direction.

In the next section we review the case of a monochromatic plane e/m wave in empty space. A more general, non-monochromatic treatment of the plane-wave-propagation problem in empty space is then described and this approach is extended to plane-wave solutions for a conducting medium. An interesting difference from the monochromatic case is noted in this regard.

Monochromatic wave in empty space

In empty space where no charges or currents (free or bound) exist, the Maxwell equations are a system of homogeneous linear first-order PDEs:

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &&(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &&(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\,\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. This system may be viewed as a Bäcklund transformation (BT; see Appendix) relating solutions of the second-order wave equations for the electric and the magnetic field. This means that the self-consistency of the system (1) requires that each field $\vec{E}$ and $\vec{B}$ satisfy a corresponding wave equation. The wave equations for $\vec{E}$ and $\vec{B}$ are thus consistency conditions for the BT (1). This can be proven by applying the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}$$

to the system (1), by which process we obtain separate wave equations for $\vec{E}$ and $\vec{B}$:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{2}$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{3}$$

where c is the speed of propagation of an e/m wave in vacuum, equal to

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{4}$$

As mentioned earlier, the fields $\vec{E}$ and $\vec{B}$ related by (1) are *conjugate* through the BT (1). To find pairs of conjugate fields we seek parametric solutions of the PDEs (2) and (3) (see Appendix). To this end we try monochromatic plane-wave solutions of angular frequency ω, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \exp\{\, i\,(\vec{k}\cdot\vec{r} - \omega t)\} \qquad (a) \\ \vec{B}(\vec{r},t) &= \vec{B}_0 \exp\{\, i\,(\vec{k}\cdot\vec{r} - \omega t)\} \qquad (b) \end{aligned} \tag{5}$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes and where

$$\frac{\omega}{k} = c \qquad (k = |\vec{k}\,|) \tag{6}$$

The ω, $\vec{k}$, $\vec{E}_0$ and $\vec{B}_0$ are the "parameters" on which the test solutions (5) depend.

The general solutions (5) are not *a priori* conjugate through the BT (1) and hence do not represent an e/m field. To find the extra constraints that must be satisfied by the parameters we need to substitute Eqs. (5) into the Maxwell system (1). By taking into account that $\vec{\nabla} e^{\,i\vec{k}\cdot\vec{r}} = i\vec{k}\, e^{\,i\vec{k}\cdot\vec{r}}$, the *div* equations (1*a*) and (1*b*) yield

$$\vec{k}\cdot\vec{E}_0 = 0 \quad (a) \qquad \vec{k}\cdot\vec{B}_0 = 0 \quad (b) \tag{7}$$

while the *rot* equations (1*c*) and (1*d*) give

$$\vec{k}\times\vec{E}_0 = \omega\,\vec{B}_0 \quad (a) \qquad \vec{k}\times\vec{B}_0 = -\frac{\omega}{c^2}\,\vec{E}_0 \quad (b) \tag{8}$$

Now, we notice that the four equations (7)–(8) do not form an independent set since (7*b*) and (8*b*) can be reproduced by using (7*a*) and (8*a*). Indeed, taking the dot product of (8*a*) with $\vec{k}$ we get (7*b*), while taking the cross product of (8*a*) with $\vec{k}$, and using (7*a*) and (6), we find (8*b*).

So, from 4 independent Maxwell equations we obtained only 2 independent pieces of information. This happened because we "fed" our trial solutions (5) with more information than necessary, in anticipation of results that follow *a posteriori* from Maxwell's equations. Thus, we assumed from the outset that the two waves (electric and magnetic) have similar simple functional forms and propagate in the same direction. By relaxing these initial assumptions our analysis acquires a richer and more interesting structure.

Non-monochromatic wave in empty space

Let us assume, more generally, that the fields $\vec{E}$ and $\vec{B}$ represent plane waves propagating in empty space in the directions of the unit vectors $\hat{\tau}$ and $\hat{\kappa}$, respectively:

$$\vec{E}(\vec{r},t) = \vec{F}(\hat{\tau}\cdot\vec{r} - ct)\ , \quad \vec{B}(\vec{r},t) = \vec{G}(\hat{\kappa}\cdot\vec{r} - ct) \tag{9}$$

Furthermore, we assume that the functions $\vec{F}$ and $\vec{G}$ can be expressed as linear combinations of monochromatic plane waves of the form (5), for continuously varying values of k and ω, where $\omega = ck$, according to (6). Then $\vec{E}$ and $\vec{B}$ can be written in Fourier-integral form, as follows:

$$\begin{aligned} \vec{E} &= \int \vec{E}_0(k)\, e^{ik(\hat{\tau}\cdot\vec{r} - ct)} dk \\ \vec{B} &= \int \vec{B}_0(k)\, e^{ik(\hat{\kappa}\cdot\vec{r} - ct)} dk \end{aligned} \tag{10}$$

In general, the integration variable k is assumed to run from 0 to $+\infty$. For notational economy, the limits of integration with respect to k will not be displayed explicitly.

By setting

$$u = \hat{\tau}\cdot\vec{r} - ct\ , \quad v = \hat{\kappa}\cdot\vec{r} - ct \tag{11}$$

we write

$$\begin{aligned} \vec{E}(u) &= \int \vec{E}_0(k)\, e^{iku} dk \\ \vec{B}(v) &= \int \vec{B}_0(k)\, e^{ikv} dk \end{aligned} \tag{12}$$

We note that

$$\vec{\nabla} e^{iku} = ik\hat{\tau} e^{iku}\ , \quad \vec{\nabla} e^{ikv} = ik\hat{\kappa} e^{ikv} \tag{13}$$

By using (12) and (13) we find that

$$\vec{\nabla}\cdot\vec{E} = \int ik\hat{\tau}\cdot\vec{E}_0(k)\, e^{iku} dk\ , \quad \vec{\nabla}\cdot\vec{B} = \int ik\hat{\kappa}\cdot\vec{B}_0(k)\, e^{ikv} dk\ ,$$

$$\vec{\nabla}\times\vec{E} = \int ik\hat{\tau}\times\vec{E}_0(k)\, e^{iku} dk\ , \quad \vec{\nabla}\times\vec{B} = \int ik\hat{\kappa}\times\vec{B}_0(k)\, e^{ikv} dk\ .$$

Moreover, we have that

$$\frac{\partial\vec{E}}{\partial t} = -\int i\omega\vec{E}_0(k)\, e^{iku} dk\ , \quad \frac{\partial\vec{B}}{\partial t} = -\int i\omega\vec{B}_0(k)\, e^{ikv} dk$$

where, as always, $\omega = ck$.

The two Gauss' laws (1*a*) and (1*b*) yield

$$\int k\hat{\tau}\cdot\vec{E}_0(k)e^{iku}dk=0 \quad \text{and} \quad \int k\hat{\kappa}\cdot\vec{B}_0(k)e^{ikv}dk=0,$$

respectively. In order that these relations be valid identically for all u and all v, respectively, we must have

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\kappa}\cdot\vec{B}_0(k)=0, \quad \text{for all } k \tag{14}$$

From Faraday's law (1*c*) and the Ampère-Maxwell law (1*d*) we obtain two more integral equations:

$$\int k\hat{\tau}\times\vec{E}_0(k)e^{iku}dk=\int \omega\vec{B}_0(k)e^{ikv}dk \tag{15}$$

$$\int k\hat{\kappa}\times\vec{B}_0(k)e^{ikv}dk=-\int\frac{\omega}{c^2}\vec{E}_0(k)e^{iku}dk \tag{16}$$

where we have taken into account Eq. (4).

Taking the cross product of (15) with $\hat{\kappa}$ and using (16), we find the integral relation

$$\int k[(\hat{\kappa}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\kappa}\cdot\hat{\tau})\vec{E}_0]e^{iku}dk=-\int k\vec{E}_0\,e^{iku}dk\,.$$

This is true for all u if

$$(\hat{\kappa}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\kappa}\cdot\hat{\tau})\vec{E}_0=-\vec{E}_0 \;\Rightarrow\; (\hat{\kappa}\cdot\hat{\tau}-1)\vec{E}_0=(\hat{\kappa}\cdot\vec{E}_0)\,\hat{\tau}\,.$$

Given that, by (14), $\vec{E}_0$ and $\hat{\tau}$ are mutually perpendicular, the above relation can only be valid if $\hat{\kappa}\cdot\hat{\tau}=1$ and $\hat{\kappa}\cdot\vec{E}_0=0$. This, in turn, can only be satisfied if $\hat{\kappa}=\hat{\tau}$. The same conclusion is reached by taking the cross product of (16) with $\hat{\tau}$ and by using (15) as well as the fact that $\vec{B}_0$ is normal to $\hat{\kappa}$. From (11) we then have that

$$u=v=\hat{\tau}\cdot\vec{r}-ct$$

so that relations (12) become

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)e^{iku}dk=\int\vec{E}_0(k)e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)e^{iku}dk=\int\vec{B}_0(k)e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\end{aligned} \tag{17}$$

Equations (14) are now rewritten as

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0, \quad \text{for all } k \tag{18}$$

Furthermore, in order that (15) and (16) (with u and $\hat{\tau}$ in place of v and $\hat{\kappa}$, respectively) be identically valid for all u, we must have

$$k\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k)\ \Leftrightarrow\ \hat{\tau}\times\vec{E}_0(k)=c\,\vec{B}_0(k) \tag{19}$$

and

$$k\,\hat{\tau}\times\vec{B}_0(k)=-\frac{\omega}{c^2}\vec{E}_0(k)\ \Leftrightarrow\ \hat{\tau}\times\vec{B}_0(k)=-\frac{1}{c}\vec{E}_0(k) \tag{20}$$

for all k, where $k=\omega/c$. Notice, however, that (19) and (20) are not independent equations, since (20) is essentially the cross product of (19) with $\hat{\tau}$.

The general plane-wave solutions to the Maxwell system (1) are thus given by relations (17) with the additional constraints (18) and (19) on the parameters. Let us summarize our main findings:

1. The fields $\vec{E}$ and $\vec{B}$ are plane waves traveling in the same direction, defined by the unit vector $\hat{\tau}$; these fields satisfy the Maxwell equations in empty space.

2. The e/m wave $(\vec{E},\vec{B})$ is a *transverse* wave. Indeed, from equations (17) and the orthogonality relations (18) it follows that

$$\hat{\tau}\cdot\vec{E}=0\quad\text{and}\quad\hat{\tau}\cdot\vec{B}=0 \tag{21}$$

3. The fields $\vec{E}$ and $\vec{B}$ are mutually perpendicular. Moreover, the vectors $(\vec{E},\vec{B},\hat{\tau})$ define a right-handed rectangular system. Indeed, by cross-multiplying (17) with $\hat{\tau}$ and by using (19) and (20), we find:

$$\hat{\tau}\times\vec{E}=c\vec{B}\ ,\quad\hat{\tau}\times\vec{B}=-\frac{1}{c}\vec{E} \tag{22}$$

4. Taking *real values* of (21) and (22), we have:

$$\hat{\tau}\cdot\mathrm{Re}\vec{E}=0\ ,\quad\hat{\tau}\cdot\mathrm{Re}\vec{B}=0\quad\text{and}\quad\hat{\tau}\times\mathrm{Re}\vec{E}=c\,\mathrm{Re}\vec{B} \tag{23}$$

The magnitude of the last vector equation in (23) gives a relation between the instantaneous values of the electric and the magnetic field:

$$|\,\mathrm{Re}\vec{E}\,|=c\,|\,\mathrm{Re}\vec{B}\,| \tag{24}$$

The above results for empty space can be extended in a straightforward way to the case of a *linear, non-conducting, non-dispersive* medium upon replacement of ε_0 and μ_0 with ε and μ, respectively (see, e.g., Papachristou, 2020). The (frequency-independent) speed of propagation of the plane e/m wave in this case is $v=1/(\varepsilon\mu)^{1/2}$.

Plane wave in a conducting medium

The Maxwell equations for a conducting medium of conductivity σ may be written as follows (Griffiths, 2013; Papachristou, 2020):

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad (c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad (d)\quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{25}$$

By using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}$$

the relations (25) lead to the *modified wave equations*

$$\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \tag{26}$$

$$\nabla^2\vec{B}-\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\frac{\partial\vec{B}}{\partial t}=0 \tag{27}$$

Guided by our monochromatic-wave approach to the problem (see, e.g., Papachristou and Magoulas, 2016) we now try a more general, integral form of solution of the above wave equations:

$$\begin{aligned} \vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\hat{\tau}\cdot\vec{r}-\omega t)}\,dk=\int\vec{E}_0(k)\exp\{(ik-s)\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk \\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\hat{\tau}\cdot\vec{r}-\omega t)}\,dk=\int\vec{B}_0(k)\exp\{(ik-s)\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk \end{aligned} \tag{28}$$

where s is a real parameter related to the conductivity of the medium. As in the vacuum case, the unit vector $\hat{\tau}$ indicates the direction of propagation of the wave. Notice that we have assumed from the outset that both waves – electric and magnetic – propagate in the same direction, in view of the fact that our results must agree with those for a non-conducting medium (in particular, for the vacuum) upon setting s=0.

It is convenient to set

$$\exp\{(ik-s)\hat{\tau}\cdot\vec{r}-i\omega t\}\equiv A(\vec{r},t) \tag{29}$$

Then, Eq. (28) takes on the form

$$\begin{aligned} \vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,A(\vec{r},t)\,dk \\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,A(\vec{r},t)\,dk \end{aligned} \tag{30}$$

The following relations can be proven:

$$\vec{\nabla} A(\vec{r},t) = (ik - s)\,\hat{\tau} A(\vec{r},t) \tag{31}$$

$$\nabla^2 A(\vec{r},t) = (s^2 - k^2 - 2isk) A(\vec{r},t) \tag{32}$$

Moreover,

$$\frac{\partial}{\partial t} A(\vec{r},t) = -i\omega A(\vec{r},t) \quad \text{and} \quad \frac{\partial^2}{\partial t^2} A(\vec{r},t) = -\omega^2 A(\vec{r},t)\,.$$

From (26) we get

$$\int [(s^2 - k^2 + \varepsilon\mu\omega^2) + i(\mu\sigma\omega - 2sk)]\, \vec{E}_0(k) A(\vec{r},t)\, dk = 0$$

[a similar integral relation is found from (27)]. This will be identically satisfied for all $\vec{r}$ and t if

$$s^2 - k^2 + \varepsilon\mu\omega^2 = 0 \quad \text{and} \quad \mu\sigma\omega - 2sk = 0 \tag{33}$$

By using relations (33), ω and s can be expressed as functions of k, as required in order that the integral relations (28) make sense. Notice, in particular, that, by the second relation (33), $s=0$ if $\sigma=0$ (non-conducting medium). Then, by the first relation, $\omega/k = 1/(\varepsilon\mu)^{1/2}$, which is the familiar expression for the speed of propagation of an e/m wave in a non-conducting medium.

From the two Gauss' laws (25*a*) and (25*b*) we get the corresponding integral relations

$$\int (ik - s)\,\hat{\tau} \cdot \vec{E}_0(k) A(\vec{r},t)\, dk = 0\,,$$
$$\int (ik - s)\,\hat{\tau} \cdot \vec{B}_0(k) A(\vec{r},t)\, dk = 0\,.$$

These will be identically satisfied for all $\vec{r}$ and t if

$$\hat{\tau} \cdot \vec{E}_0(k) = 0 \quad \text{and} \quad \hat{\tau} \cdot \vec{B}_0(k) = 0, \ \text{for all } k \tag{34}$$

From (25*c*) and (25*d*) we find

$$\int (ik - s)\,\hat{\tau} \times \vec{E}_0(k) A(\vec{r},t)\, dk = \int i\omega \vec{B}_0(k) A(\vec{r},t)\, dk$$

and

$$\int (ik - s)\,\hat{\tau} \times \vec{B}_0(k) A(\vec{r},t)\, dk = \int (\mu\sigma - i\varepsilon\mu\omega) \vec{E}_0(k) A(\vec{r},t)\, dk\,,$$

respectively. To satisfy these for all $\vec{r}$ and t, we require that

$$(k + is)\,\hat{\tau} \times \vec{E}_0(k) = \omega \vec{B}_0(k) \tag{35}$$

and

$$(k+is)\,\hat{\tau}\times\vec{B}_0(k) = -(\varepsilon\mu\omega + i\mu\sigma)\vec{E}_0(k) \tag{36}$$

Note, however, that (36) is not an independent equation since it can be reproduced by cross-multiplying (35) with $\hat{\tau}$ and by taking into account Eqs. (33) and (34).

We note the following:

1. From (30) and (34) we have that

$$\hat{\tau}\cdot\vec{E} = 0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B} = 0 \tag{37}$$

or, in real form, $\hat{\tau}\cdot\mathrm{Re}\,\vec{E} = 0$ and $\hat{\tau}\cdot\mathrm{Re}\,\vec{B} = 0$. This means that both $\mathrm{Re}\,\vec{E}$ and $\mathrm{Re}\,\vec{B}$ are normal to the direction of propagation of the wave.

2. From (30) and (35) we get

$$\hat{\tau}\times\vec{E} = \int \frac{\omega}{k+is}\vec{B}_0(k)A(\vec{r},t)\,dk \tag{38}$$

The integral on the right-hand side of (38) is, generally, not a vector parallel to $\vec{B}$. Now, in the limit of negligible conductivity (σ=0) the relations (33) give s=0 and ω/k=1/($\varepsilon\mu$)$^{1/2}$. The quotient ω/k represents the speed of propagation υ in the non-conducting medium, for the frequency ω. If the medium is *non-dispersive*, the speed $\upsilon=\omega/k$ is constant, independent of frequency. Then Eq. (38) (with s=0) becomes

$$\hat{\tau}\times\vec{E} = \upsilon\int\vec{B}_0(k)A(\vec{r},t)\,dk = \upsilon\vec{B}$$

and, in real form, it reads $\hat{\tau}\times\mathrm{Re}\vec{E} = \upsilon\,\mathrm{Re}\vec{B}$. Geometrically, this means that the vectors $(\mathrm{Re}\,\vec{E}, \mathrm{Re}\,\vec{B}, \hat{\tau})$ define a right-handed rectangular system.

3. The vectors $\vec{E}$ and $\vec{B}$ are mutually perpendicular in a *monochromatic* e/m wave of definite frequency ω, traveling in a conducting medium (see, e.g., Papachristou and Magoulas, 2016). Such a wave is represented in real form by the equations

$$\vec{E}(\vec{r},t) = \vec{E}_0\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\alpha)\,,$$
$$\vec{B}(\vec{r},t) = \frac{\sqrt{k^2+s^2}}{\omega}(\hat{\tau}\times\vec{E}_0)\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\beta)$$

where $\vec{E}_0$ is a real vector and where $\tan(\beta-\alpha)=s/k$. This perpendicularity between $\vec{E}$ and $\vec{B}$ ceases to exist, however, in a non-monochromatic wave of the form (28).

Summary

The Maxwell equations of electrodynamics may be viewed as a Bäcklund transformation (BT) relating solutions of the wave equations for the electric and the magnetic field (Papachristou, 2015; Papachristou and Magoulas, 2016). Wave solutions for the *E* and *B*-fields that jointly satisfy the BT are said to be *conjugate* through the BT.

Pairs of conjugate solutions can be found by assuming general, parameter-dependent solutions of the two wave equations (electric and magnetic) and by seeking the conditions these parameters must obey in order for the Maxwell BT to be satisfied. The usual choice of test solutions is the monochromatic plane e/m wave, the properties of which are, of course, well known.

In this article a more general choice of wave solutions was made that better exploits the independence of the Maxwell equations. It concerns the non-monochromatic plane wave composed of an infinite number of monochromatic waves of various frequencies. The non-monochromatic case exhibits some interesting properties. For example, one does not have to *a priori* assume that the electric and magnetic waves propagate in the same direction: this is dictated by the Maxwell system itself. Moreover, it is only in the monochromatic case that the *E* and *B*-fields of a plane e/m wave in a conducting medium are mutually perpendicular, whereas this is not the case for a non-monochromatic wave in such a medium (in fact, in any *dispersive* medium).

Appendix. Bäcklund transformations

Consider two PDEs $P[u]=0$ and $Q[v]=0$ for the unknown functions u and v, respectively. The expressions $P[u]$ and $Q[v]$ may contain the corresponding variables u and v, as well as partial derivatives of u and v with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, y. Partial derivatives with respect to these variables will be denoted by using subscripts: u_x, u_y, u_{xx}, u_{yy}, u_{xy}, etc.

Independently, for the moment, also consider a pair of coupled PDEs for u and v:

$$B_1[u,v]=0 \quad (a) \qquad B_2[u,v]=0 \quad (b) \tag{39}$$

where the expressions $B_i[u,v]$ $(i=1,2)$ may contain u, v as well as partial derivatives of u and v with respect to x and y. We note that u appears in both equations (a) and (b). The question then is: if we find an expression for u by integrating (a) for a given v, will it match the corresponding expression for u found by integrating (b) for the same v? The answer is that, in order that (a) and (b) be consistent with each other for solution for u, the function v must be properly chosen so as to satisfy a certain *consistency condition* (or *integrability condition* or *compatibility condition*).

By a similar reasoning, in order that (a) and (b) in (39) be mutually consistent for solution for v, for some given u, the function u must now itself satisfy a corresponding integrability condition.

If it happens that the two consistency conditions for integrability of the system (39) are precisely the PDEs $P[u]=0$ and $Q[v]=0$, we say that this system constitutes a *Bäcklund transformation* (BT) connecting solutions of $P[u]=0$ with solutions of

$Q[v]=0$. In the special case where $P\equiv Q$, i.e., if u and v satisfy *the same* PDE, the system (39) is called an *auto-Bäcklund* transformation (auto-BT) for this PDE.

Suppose now that we seek solutions of the PDE $P[u]=0$. Assume that we are able to find a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$, the auto-BT connects solutions u and v of the same PDE) and let $v=v_0(x,y)$ be some known solution of $Q[v]=0$. The BT is then a system of PDEs for the unknown u,

$$B_i[u, v_0] = 0 \ , \quad i = 1, 2 \tag{40}$$

The system (40) is integrable for u, given that the function v_0 satisfies *a priori* the required integrability condition $Q[v]=0$. The solution u then of the system satisfies the PDE $P[u]=0$. Thus a solution $u(x,y)$ of the latter PDE is found without actually solving the equation itself, simply by integrating the BT (40) with respect to u. Of course, this method will be useful provided that integrating the system (40) for u is simpler than integrating the PDE $P[u]=0$ itself. If the transformation (40) is an auto-BT for the PDE $P[u]=0$, then, starting with a known solution $v_0(x,y)$ of this equation and integrating the system (40) we find another solution $u(x,y)$ of the same equation.

Let us see an example of the use of a BT to generate solutions of a PDE. The Cauchy-Riemann relations of Complex Analysis,

$$u_x = v_y \quad (a) \qquad u_y = -v_x \quad (b) \tag{41}$$

constitute an auto-BT for the Laplace equation,

$$P[w] \equiv w_{xx} + w_{yy} = 0 \tag{42}$$

Let us explain this: Suppose we want to solve the system (41) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (a) and (b) match for solution for u we must compare them in some way. We thus differentiate (a) with respect to y and (b) with respect to x and equate the mixed derivatives of u. That is, we apply the integrability condition $(u_x)_y = (u_y)_x$. In this way we eliminate the variable u and find the condition that must be obeyed by $v(x,y)$:

$$P[v] \equiv v_{xx} + v_{yy} = 0 \ .$$

Similarly, by using the integrability condition $(v_x)_y = (v_y)_x$ to eliminate v from the system (41), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$P[u] \equiv u_{xx} + u_{yy} = 0 \ .$$

In conclusion, the integrability of the system (41) with respect to either variable requires that the other variable satisfy the Laplace equation (42).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (42). Substituting $v=v_0$ in the system (41), we can integrate this system with respect to u. It is not hard to show (by eliminating v_0 from the system) that the solution u will also satisfy the

Laplace equation (42). As an example, by choosing the solution $v_0(x,y)=xy$, we find a new solution $u(x,y)= (x^2 - y^2)/2 + C$.

As presented above, a BT is an auxiliary device for constructing solutions of a PDE from known solutions of the same or another PDE (this method is particularly useful for finding solutions of nonlinear PDEs). A related problem, where solutions of the differential system representing the BT itself are sought, is also of interest. To be specific, assume that we need to integrate a given system of PDEs connecting two functions u and v:

$$B_i[u,v]=0\ , \quad i=1,2 \tag{43}$$

Suppose that the integrability of the system for both functions requires that u and v separately satisfy the respective PDEs

$$P[u]=0 \quad (a) \qquad Q[v]=0 \quad (b) \tag{44}$$

That is, the system (43) is a BT connecting solutions of the PDEs (44). Assume, now, that the latter PDEs possess known (or, in any case, easy to find) parameter-dependent solutions of the form

$$u = f(x,y;\alpha,\beta,\ldots)\ , \quad v = g(x,y;\kappa,\lambda,\ldots)$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which u and v jointly satisfy the system (43), we say that the solutions u and v of the PDEs (44*a*) and (44*b*), respectively, are *conjugate* through the BT (43) (or BT-conjugate, for short). By finding a pair of BT-conjugate solutions one thus automatically obtains a solution of the system (43).

Note that parametric solutions of *both* integrability conditions $P[u]=0$ and $Q[v]=0$ must be known in advance! From the practical point of view the method is thus most applicable in linear problems, since it is much easier to find parameter-dependent solutions of the PDEs (44) in this case.

Let us see an example: Going back to the Cauchy-Riemann relations (41), we try the following parametric solutions of the Laplace equation (42):

$$u(x,y)=\alpha(x^2-y^2)+\beta x+\gamma y\ ,$$
$$v(x,y)=\kappa xy+\lambda x+\mu y\ .$$

Substituting these into the BT (41), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$u(x,y)=\alpha(x^2-y^2)+\beta x+\gamma y\ ,$$
$$v(x,y)=2\alpha xy-\gamma x+\beta y$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u(x,y)=\alpha xy\ , \quad v(x,y)=\beta xy\ .$$

Inserting these into the system (41) and taking into account the independence of x and y, we find that the only possible values of the parameters α and β are $\alpha=\beta=0$, so that $u(x,y)= v(x,y)=0$. Thus, no non-trivial BT-conjugate solutions exist in this case.

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On active and passive transformations

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The concepts of active and passive transformations on a vector space are discussed. Orthogonal coordinate transformations and matrix representations of linear operators are considered in particular.

1. Introduction

A physical situation may *appear* changing for two reasons: the physical system itself may pass from one state to another, or, the *same* state of the system may be viewed from two different points of view (e.g., by two different observers, using different frames of reference). The former case corresponds to an *"active"* view of the situation, while the latter one to a *"passive"* view.

Given that many physical quantities are vectors, of particular interest in Physics are linear transformations on vector spaces. Starting with the prototype transformation of rotation on a plane, we study both the active and the passive view of these transformations. In the case of a Euclidean space with Cartesian coordinates, a passive transformation corresponding to a change of basis is an orthogonal transformation. On the other hand, an active transformation on a vector space is produced by a linear operator, which is represented by a matrix in a given basis. A change of basis, leading to a different representation, is a passive transformation on this space.

2. Active view of transformations

Consider the *xy*-plane with Cartesian coordinates (x, y) and basis unit vectors $\{\hat{u}_x, \hat{u}_y\}$. We call $\mathbf{R}(\theta)$ the rotation operator on this plane, i.e., the operator which rotates any vector $\vec{A}$ on the plane by an angle θ (see Fig. 2.1; by convention, $\theta>0$ for counterclockwise rotation while $\theta<0$ for clockwise rotation). This operator is linear, given that adding two vectors and then rotating the sum is the same as first rotating the vectors and then adding them.

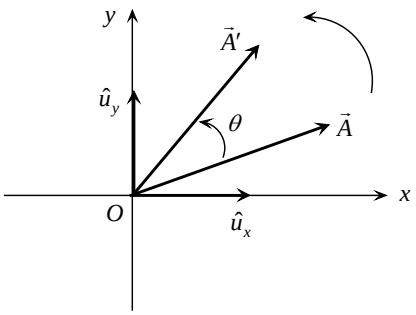

Figure 2.1

Imagine, in particular, that we rotate each vector in the basis $\{\hat{u}_x, \hat{u}_y\}$ by an angle θ to obtain a new set of vectors $\{\hat{u}_x', \hat{u}_y'\}$ (Fig. 2.2). The transformation equations describing these rotations are

$$\begin{aligned} \hat{u}_x' &= \mathbf{R}(\theta)\hat{u}_x = \cos\theta\,\hat{u}_x + \sin\theta\,\hat{u}_y \\ \hat{u}_y' &= \mathbf{R}(\theta)\hat{u}_y = -\sin\theta\,\hat{u}_x + \cos\theta\,\hat{u}_y \end{aligned} \tag{2.1}$$

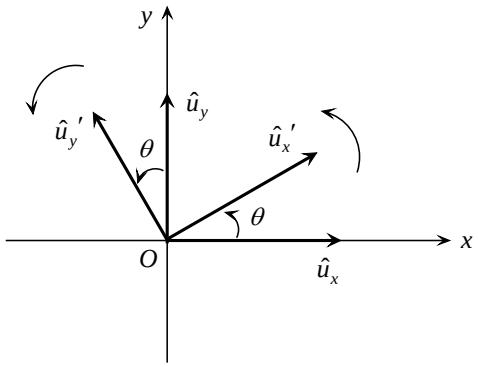

Figure 2.2

Now, let $\vec{A} = A_x\hat{u}_x + A_y\hat{u}_y$ be a vector on the *xy*-plane (see Fig. 2.1). The rotation operator $\mathbf{R}(\theta)$ will transform it into a new vector

$$\vec{A}' = \mathbf{R}(\theta)\vec{A} = A_x'\hat{u}_x + A_y'\hat{u}_y \tag{2.2}$$

We want to express the components A_x' and A_y' in terms of A_x, A_y and θ. By the linearity of $\mathbf{R}(\theta)$ and by using (2.1), we have:

$$\begin{aligned} \vec{A}' &= \mathbf{R}(\theta)\left(A_x\hat{u}_x + A_y\hat{u}_y\right) = A_x\mathbf{R}(\theta)\hat{u}_x + A_y\mathbf{R}(\theta)\hat{u}_y \\ &= \left(A_x\cos\theta - A_y\sin\theta\right)\hat{u}_x + \left(A_x\sin\theta + A_y\cos\theta\right)\hat{u}_y \end{aligned}$$

By comparing this with (2.2), we get:

$$\begin{aligned} A_x' &= A_x\cos\theta - A_y\sin\theta \\ A_y' &= A_x\sin\theta + A_y\cos\theta \end{aligned} \tag{2.3}$$

We define the matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.4}$$

The systems (2.1) and (2.3) are then rewritten in the form of matrix equations as

$$\begin{bmatrix} \hat{u}_x' \\ \hat{u}_y' \end{bmatrix} = M^T \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_x' \\ A_y' \end{bmatrix} = M \begin{bmatrix} A_x \\ A_y \end{bmatrix} \tag{2.5}$$

respectively, where M^T is the transpose of M.

We note that the vectors $\vec{A}$ and $\vec{A}' = \mathbf{R}(\theta)\vec{A}$ are *different* geometrical objects, the latter one being a transformation of the former. On the other hand, the components of these vectors, connected by (2.3), are referred to the *same* basis $\{\hat{u}_x, \hat{u}_y\}$. This is the general idea of the *active view* of a linear transformation.

In a more abstract sense, we consider an n-dimensional vector space Ω with basis vectors $\{\hat{e}_1, \hat{e}_2, ..., \hat{e}_n\} \equiv \{\hat{e}_k\}$, and we let **R** be a linear operator on Ω. We assume that the basis vectors transform under **R** as follows:

$$\hat{e}_i' = \mathbf{R}\,\hat{e}_i = \hat{e}_j R^j{}_i \quad \text{(sum on } j\text{)} \tag{2.6}$$

where the familiar summation convention for repeated upper and lower indices has been used. Thus, for each value of i, the right-hand side of (2.6) is actually a sum over all values of j, i.e., from j=1 to j=n. Explicitly,

$$\begin{aligned} \hat{e}_1' &= \hat{e}_1 R^1{}_1 + \hat{e}_2 R^2{}_1 + \cdots + \hat{e}_n R^n{}_1 \\ \hat{e}_2' &= \hat{e}_1 R^1{}_2 + \hat{e}_2 R^2{}_2 + \cdots + \hat{e}_n R^n{}_2 \\ &\vdots \\ \hat{e}_n' &= \hat{e}_1 R^1{}_n + \hat{e}_2 R^2{}_n + \cdots + \hat{e}_n R^n{}_n \end{aligned} \tag{2.7}$$

Now, let

$$\vec{V} = V^1\hat{e}_1 + V^2\hat{e}_2 + \cdots + V^n\hat{e}_n \equiv V^i\hat{e}_i \tag{2.8}$$

be a vector in Ω, and let $\vec{V}' = \mathbf{R}\vec{V}$. We have:

$$\vec{V}' = \mathbf{R}(V^j\hat{e}_j) = V^j\mathbf{R}\,\hat{e}_j = V^j\hat{e}_i R^i{}_j \equiv V^{i\prime}\hat{e}_i \ .$$

Therefore the components of the original and the transformed vector are related by

$$V^{i\prime} = R^i{}_j V^j \tag{2.9}$$

or, explicitly,

$$\begin{aligned} V^{1\prime} &= R^1{}_1 V^1 + R^1{}_2 V^2 + \cdots + R^1{}_n V^n \\ V^{2\prime} &= R^2{}_1 V^1 + R^2{}_2 V^2 + \cdots + R^2{}_n V^n \\ &\vdots \\ V^{n\prime} &= R^n{}_1 V^1 + R^n{}_2 V^2 + \cdots + R^n{}_n V^n \end{aligned} \tag{2.10}$$

Define the $n \times n$ matrix

$$M = \left[R^i{}_j \right] \quad \text{with} \quad M_{ij} = R^i{}_j \tag{2.11}$$

The basis transformations (2.6) are then written as

$$\begin{bmatrix} \hat{e}_1{}' \\ \vdots \\ \hat{e}_n{}' \end{bmatrix} = M^T \begin{bmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{bmatrix} \tag{2.12}$$

while the component transformations (2.9) become

$$\begin{bmatrix} V^{1\prime} \\ \vdots \\ V^{n\prime} \end{bmatrix} = M \begin{bmatrix} V^1 \\ \vdots \\ V^n \end{bmatrix} \tag{2.13}$$

3. Passive view of transformations

Imagine that our previous x-y system of axes on the plane, with basis unit vectors $\{\hat{u}_x, \hat{u}_y\}$, is rotated counterclockwise by an angle θ to obtain a new system of axes x' and y' with corresponding basis $\{\hat{u}_x{}', \hat{u}_y{}'\}$ (Fig. 3.1). As before, the two bases are related by the system of equations

$$\begin{aligned} \hat{u}_x{}' &= \cos\theta\, \hat{u}_x + \sin\theta\, \hat{u}_y \\ \hat{u}_y{}' &= -\sin\theta\, \hat{u}_x + \cos\theta\, \hat{u}_y \end{aligned} \tag{3.1}$$

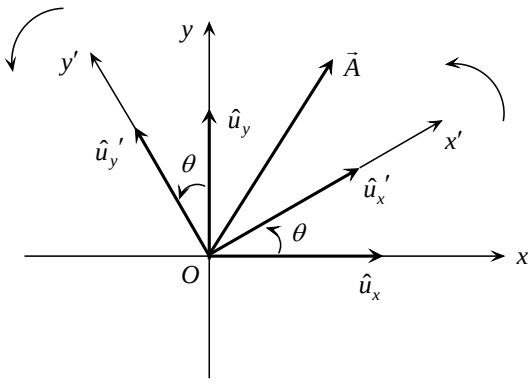

Figure 3.1

A vector $\vec{A}$ on the plane can be expressed in both these bases, as follows:

$$\vec{A} = A_x \hat{u}_x + A_y \hat{u}_y = A_x' \hat{u}_x' + A_y' \hat{u}_y' \tag{3.2}$$

Substituting the basis transformations (3.1) into the right-hand side of (3.2), and equating coefficients of similar unprimed basis vectors, we find:

$$\begin{aligned} A_x &= A_x' \cos\theta - A_y' \sin\theta \\ A_y &= A_x' \sin\theta + A_y' \cos\theta \end{aligned} \tag{3.3}$$

Solving this for the primed components, we get:

$$\begin{aligned} A_x' &= A_x \cos\theta + A_y \sin\theta \\ A_y' &= -A_x \sin\theta + A_y \cos\theta \end{aligned} \tag{3.4}$$

Notice that, in contrast to what we did in the previous section, here we keep the geometrical object $\vec{A}$ *fixed* and simply expand it in two *different* bases. This is the adopted practice in the *passive view* of a transformation.

Introducing the matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

we rewrite our previous equations in the matrix forms

$$\begin{bmatrix} \hat{u}_x' \\ \hat{u}_y' \end{bmatrix} = M^T \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \end{bmatrix} \tag{3.5}$$

and

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = M \begin{bmatrix} A_x' \\ A_y' \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} A_x' \\ A_y' \end{bmatrix} = M^{-1} \begin{bmatrix} A_x \\ A_y \end{bmatrix} \tag{3.6}$$

where

$$M^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = M^T \tag{3.7}$$

Notice that the transformation matrix *M* is *orthogonal*. As will be shown below, this is related to the fact that the transformation (rotation of axes) relates two Cartesian bases in a Euclidean space.

By comparing (2.3) and (3.4) it follows that the transformation equations of the passive view reduce to those of the active view upon replacing θ with $-\theta$. Physically this means that a passive transformation in which the vector $\vec{A}$ is fixed and the basis of our space is rotated *counterclockwise* is equivalent to an active transformation in which the basis is fixed and the vector $\vec{A}$ is rotated *clockwise*.

Let us generalize to the case of an n-dimensional vector space Ω with basis $\{\hat{e}_1, \hat{e}_2, ..., \hat{e}_n\} \equiv \{\hat{e}_k\}$. Let $\{\hat{e}_k{}'\}$ be another basis related to the former one by

$$\hat{e}_i{}' = \hat{e}_j \Lambda^j{}_{i'} \tag{3.8}$$

(note sum on j). A vector $\vec{V}$ in Ω may be expressed in both these bases, as follows:

$$\vec{V} = V^i \hat{e}_i = V^{j'} \hat{e}_j{}' = V^{j'} \hat{e}_i \Lambda^i{}_{j'}$$

where use has been made of (3.8). This yields

$$V^i = \Lambda^i{}_{j'} V^{j'} \tag{3.9}$$

Introducing the $n \times n$ matrix

$$M = \left[\Lambda^i{}_{j'} \right] \quad \text{with} \quad M_{ij} = \Lambda^i{}_{j'} \tag{3.10}$$

we write

$$\begin{bmatrix} \hat{e}_1{}' \\ \vdots \\ \hat{e}_n{}' \end{bmatrix} = M^T \begin{bmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{bmatrix} \tag{3.11}$$

and

$$\begin{bmatrix} V^1 \\ \vdots \\ V^n \end{bmatrix} = M \begin{bmatrix} V^{1'} \\ \vdots \\ V^{n'} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} V^{1'} \\ \vdots \\ V^{n'} \end{bmatrix} = M^{-1} \begin{bmatrix} V^1 \\ \vdots \\ V^n \end{bmatrix} \tag{3.12}$$

4. Orthogonal transformations in a Euclidean space

In this section the *passive* view of transformations will be adopted. Let Ω be an n-dimensional Euclidean space with Cartesian[1] coordinates $(x^1, x^2, ..., x^n) \equiv (x^k)$ and corresponding Cartesian basis $\{\hat{e}_k\}$. Let $(x^{k'})$ be another Cartesian coordinate system for

[1] Cartesian systems of coordinates exist only in Euclidean spaces. For example, you can define a system of Cartesian coordinates on a plane but you *cannot* define such coordinates on the surface of a sphere, which is a *non-Euclidean* space.

Ω, with corresponding basis $\{\hat{e}_k{}'\}$. We assume that the two coordinate systems have a common origin $O\equiv(0,0,...,0)$. Both Cartesian bases are *orthonormal*, in the sense that

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}_i{}' \cdot \hat{e}_j{}' = \delta_{ij} \tag{4.1}$$

Assuming that the *handedness* of the two coordinate systems is the same (e.g., for n=3, both coordinate systems are right-handed) it is apparent that a linear transformation from one basis to the other is a "rotation" in Ω. Let us explore this in more detail.

Definition: A linear transformation from a Cartesian basis to another is said to be an *orthogonal transformation*.

Proposition 4.1: An orthogonal transformation is represented by an *orthogonal* matrix M:

$$M^{-1} = M^T \quad \Leftrightarrow \quad M^T M = M M^T = \mathbf{1} \tag{4.2}$$

Proof: Assume a linear basis transformation of the form (3.8): $\hat{e}_i{}' = \hat{e}_j \Lambda^j{}_{i'}$. Also, let M be the transformation matrix defined in (3.10). We have:

$$\hat{e}_i{}' \cdot \hat{e}_j{}' = \left(\hat{e}_k \Lambda^k{}_{i'}\right) \cdot \left(\hat{e}_l \Lambda^l{}_{j'}\right) = \delta_{kl} \Lambda^k{}_{i'} \Lambda^l{}_{j'} = \sum_k \Lambda^k{}_{i'} \Lambda^k{}_{j'}$$
$$= \sum_k M_{ki} M_{kj} = \sum_k \left(M^T\right)_{ik} M_{kj} = \left(M^T M\right)_{ij}$$

where we have taken into account that the original (unprimed) basis is orthonormal. Given that the same is true for the transformed (primed) basis, we have:

$$\left(M^T M\right)_{ij} = \delta_{ij} \quad \Leftrightarrow \quad M^T M = \mathbf{1}.$$

The *magnitude* of a vector $\vec{V}$ is a non-negative quantity whose square is expressed in a Cartesian basis in terms of the scalar (dot) product, as follows:

$$\left|\vec{V}\right|^2 = \vec{V} \cdot \vec{V} = \left(V^i \hat{e}_i\right) \cdot \left(V^j \hat{e}_j\right) = V^i V^j \hat{e}_i \cdot \hat{e}_j = \delta_{ij} V^i V^j \tag{4.3}$$

[Obviously, the last term in (4.3) is the sum of the squares of the components of $\vec{V}$.]

Proposition 4.2: An orthogonal transformation preserves the Cartesian form (4.3) of the magnitude of a vector.

Proof: By using the transformation formula (3.9) for components of vectors, derived in the previous section, we have:

$$\delta_{ij}V^iV^j = \delta_{ij}\left(\Lambda^i{}_{k'}V^{k'}\right)\left(\Lambda^j{}_{l'}V^{l'}\right) = \left(\sum_i \Lambda^i{}_{k'}\Lambda^i{}_{l'}\right)V^{k'}V^{l'}$$
$$= \left(\sum_i M_{ik}M_{il}\right)V^{k'}V^{l'} = \left(\sum_i \left(M^T\right)_{ki}M_{il}\right)V^{k'}V^{l'}$$
$$= \left(M^TM\right)_{kl}V^{k'}V^{l'} = \delta_{kl}V^{k'}V^{l'}$$

For a more compact proof, define the matrices

$$\left[V^k\right] \equiv \begin{bmatrix} V^1 \\ \vdots \\ V^n \end{bmatrix} \quad \text{and} \quad \left[V^k\right]^T \equiv \begin{bmatrix} V^1 & \cdots & V^n \end{bmatrix}$$

and similarly for the corresponding primed quantities. Then, in the unprimed basis,

$$\left|\vec{V}\right|^2 = \left[V^k\right]^T\left[V^k\right].$$

Using the fact that, by (3.12), $\left[V^k\right] = M\left[V^{k'}\right]$, we have:

$$\left[V^k\right]^T\left[V^k\right] = \left(M\left[V^{k'}\right]\right)^T M\left[V^{k'}\right] = \left[V^{k'}\right]^T M^TM\left[V^{k'}\right]$$
$$= \left[V^{k'}\right]^T\left[V^{k'}\right]$$

Comment: The above proof suggests an alternate definition of an orthogonal transformation as a linear transformation in a Euclidean space that preserves the Cartesian form of the magnitude of vectors. In fact, this is the way orthogonal transformations are usually defined in textbooks.

Now, let P be a point in Ω, with Cartesian coordinates $(x^1, x^2,...,x^n) \equiv (x^k)$. In this system of coordinates the position vector of P can be written as $\vec{r} = x^i\hat{e}_i$. Since this vector is a geometrical object independent of the system of coordinates, we can write:

$$\vec{r} = x^i\hat{e}_i = x^{j'}\hat{e}_{j'}.$$

By using (3.8) we find, as in Sec. 3,

$$x^i = \Lambda^i{}_{j'}\,x^{j'} \tag{4.4}$$

which is the analog of (3.9). If M is the matrix defined in (3.10), and if $[x^k]$ is the column vector of the x^k, then by the general matrix relation (3.12) we have:

$$\left[x^k\right] = M\left[x^{k'}\right] \quad \Rightarrow \quad \left[x^{k'}\right] = M^{-1}\left[x^k\right] = M^T\left[x^k\right] \tag{4.5}$$

where the orthogonality condition (4.2) has been used. Let us call

$$M^T \equiv L \quad \text{with} \quad L_{ij} = M_{ji} = \Lambda^{j}{}_{i'} \tag{4.6}$$

Then the matrix relation (4.5) can be written as a system of n linear equations of the form

$$\begin{aligned} x^{1'} &= L_{11}x^1 + L_{12}x^2 + \cdots + L_{1n}x^n \\ x^{2'} &= L_{21}x^1 + L_{22}x^2 + \cdots + L_{2n}x^n \\ &\vdots \\ x^{n'} &= L_{n1}x^1 + L_{n2}x^2 + \cdots + L_{nn}x^n \end{aligned} \tag{4.7}$$

which equations represent an orthogonal coordinate transformation in Ω.

As an example for n=2, let Ω be a plane with Cartesian coordinates $(x^1, x^2) \equiv (x, y)$. A position vector in Ω is written: $\vec{r} = x\hat{u}_x + y\hat{u}_y$. As seen in Sec. 3, the transformation matrix M for a rotation of the basis vectors by an angle θ is

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \Rightarrow \quad L = M^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The coordinate transformation equations (4.7) are written here as

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

Exercise: By using the relations $\vec{V} = V^j\hat{e}_j$ and $\hat{e}_j{}' = \hat{e}_l\Lambda^l{}_{j'}$, together with (3.10) and (4.1), show the following:

$$V^i = \hat{e}_i \cdot \vec{V},$$

$$M_{ij} = \hat{e}_i \cdot \hat{e}_j{}'.$$

Under an orthogonal transformation from one Cartesian system of coordinates to another, the components V^k of a vector transform like the coordinates x^k themselves. That is,

$$V^{i'} = L_{ij}V^j.$$

From (4.7) we have that

$$L_{ij} = \frac{\partial x^{i'}}{\partial x^{j}} .$$

Therefore,

$$V^{i'} = \frac{\partial x^{i'}}{\partial x^{j}} V^{j} \quad \text{and, conversely,} \quad V^{i} = \frac{\partial x^{i}}{\partial x^{j'}} V^{j'} \tag{4.8}$$

5. Active and passive view combined

Let Ω be an n-dimensional vector space with basis $\{\hat{e}_k\}$ $(k = 1, 2, \ldots, n)$. Let **A** be a linear operator on Ω. The action of **A** on the basis vectors is given by

$$\mathbf{A}\,\hat{e}_j = \sum_i \hat{e}_i A_{ij} \equiv \hat{e}_i A_{ij} \tag{5.1}$$

(Note a slight change in the summation convention; in this section subscripts only will be used.) The $n \times n$ matrix $A=[A_{ij}]$ is the *matrix representation of the operator* **A** in the basis $\{\hat{e}_k\}$.

A vector in Ω is written:

$$\vec{x} = \sum_i x_i \hat{e}_i \equiv x_i \hat{e}_i \tag{5.2}$$

Let $\vec{y} = \mathbf{A}\vec{x}$. If $\vec{y} = y_i \hat{e}_i$, then, by the linearity of **A** and by using (5.1) and (5.2) we find that

$$y_i = A_{ij} x_j \quad \text{(sum on } j\text{)} \tag{5.3}$$

which represents a system of n linear equations for i=1,...,n. In matrix form,

$$[y_k] = A\,[x_k] \tag{5.4}$$

where $[x_k]$ and $[y_k]$ are column vectors.

Now, let **A** and **B** be linear operators on Ω. We define their product **C**=**AB** by

$$\mathbf{C}\vec{x} = (\mathbf{AB})\vec{x} \equiv \mathbf{A}(\mathbf{B}\vec{x}) , \quad \forall \vec{x} \in \Omega \tag{5.5}$$

Then, in the basis $\{\hat{e}_k\}$,

$$\mathbf{C}\,\hat{e}_j = \mathbf{A}(\mathbf{B}\,\hat{e}_j) = \mathbf{A}(\hat{e}_l B_{lj}) = B_{lj}\,(\mathbf{A}\,\hat{e}_l) = A_{il} B_{lj}\,\hat{e}_i \equiv \hat{e}_i C_{ij}$$

where

$$C_{ij} = A_{il} B_{lj} \quad \text{or, in matrix form,} \quad C = AB \tag{5.6}$$

That is, in any basis of Ω,

> *the matrix of the product of two operators is the product of the matrices of these operators.*

Consider now a change of basis (passive transformation) with transformation matrix $T=[T_{ij}]$:

$$\hat{e}_j{}' = \hat{e}_i T_{ij} \tag{5.7}$$

The inverse transformation is

$$\hat{e}_j = \hat{e}_i{}' \left(T^{-1}\right)_{ij} \tag{5.8}$$

The same vector may be expressed in both these bases as $\vec{x} = x_i \hat{e}_i = x_j{}' \hat{e}_j{}'$, from which we get, by using (5.7) and (5.8),

$$x_i = T_{ij}\, x_j{}' \quad \text{and} \quad x_i{}' = \left(T^{-1}\right)_{ij} x_j \tag{5.9}$$

In matrix form,

$$[x_k] = T\,[x_k{}'] \quad \text{and} \quad [x_k{}'] = T^{-1}[x_k] \tag{5.10}$$

How do the matrix elements of a linear operator **A** transform under a change of basis of the form (5.7)? In other words, how does the matrix of an active transformation transform under a passive transformation? Let $\vec{y} = \mathbf{A}\vec{x}$. By combining (5.10) with (5.4), we have:

$$[y_k{}'] = T^{-1}[y_k] = T^{-1}A[x_k] = T^{-1}AT\,[x_k{}'] \equiv A'[x_k{}'] \;\Rightarrow$$

$$A' = T^{-1} A\, T \tag{5.11}$$

For an alternative proof, note that

$$\mathbf{A}\hat{e}_j{}' = \mathbf{A}(\hat{e}_i T_{ij}) = T_{ij}\,\mathbf{A}\hat{e}_i = T_{ij}\,\hat{e}_l A_{li} = A_{li} T_{ij}\,\hat{e}_k{}' \left(T^{-1}\right)_{kl}$$

$$= \left(T^{-1}AT\right)_{kj} \hat{e}_k{}' \equiv \hat{e}_k{}' A'_{kj} \;\Rightarrow\; A' = T^{-1}AT$$

as before. A transformation of the form (5.11) is called a *similarity transformation*.

By applying the properties of the trace and the determinant of a matrix to (5.11) it is not hard to show that, under basis transformations, *the trace and the determinant of the matrix representation of an operator remain unchanged:* tr$A$=trA', det$A$=detA'. This means that the trace and the determinant are basis-independent quantities that are properties of the operator itself, rather than properties of its representation.

Definition: A vector $\vec{x} \neq 0$ is said to be an *eigenvector* of the linear operator **A** if a constant λ exists such that

$$\mathbf{A}\,\vec{x} = \lambda\,\vec{x} \tag{5.12}$$

The constant λ is an *eigenvalue* of **A**, to which eigenvalue this eigenvector belongs. Note that, in general, more than one eigenvector may belong to the same eigenvalue.

In a given basis $\{\hat{e}_k\}$, the linear system (5.3) corresponding to the eigenvalue equation (5.12) takes on the form

$$A_{ij}\,x_j = \lambda\,x_i \qquad \text{or} \qquad (A_{ij} - \lambda\,\delta_{ij})\,x_j = 0 \tag{5.13}$$

where $[A_{ij}]=A$ is the matrix of the operator **A** in the given basis. This is a homogeneous linear system of equations, which has a nontrivial solution for the eigenvector components iff

$$\det[A_{ij} - \lambda\,\delta_{ij}] = 0 \qquad \text{or} \qquad \det(A - \lambda 1) = 0 \tag{5.14}$$

where 1 here is the *n*-dimensional unit matrix. This polynomial equation determines the eigenvalues λ_i (not necessarily all different from each other) of the operator **A**.

Now, in general, for any value of the constant λ the matrix $(A–\lambda 1)$ is the representation of the operator $(\mathbf{A}–\lambda\mathbf{1})$ in the considered basis $\{\hat{e}_k\}$. Under a basis transformation to $\{\hat{e}_k{}'\}$ this matrix transforms according to (5.11):

$$(A–\lambda 1)' = T^{-1}\,(A–\lambda 1)\,T = T^{-1}A\,T – \lambda 1 \equiv A' – \lambda 1\ .$$

On the other hand, by the invariance of the determinant under this transformation,

$$\det(A' – \lambda 1) = \det(A – \lambda 1)\ .$$

In particular, if λ is an eigenvalue of the operator **A**, the right-hand side of the above equation vanishes in view of (5.14) and, therefore, the same must be true for the left-hand side *for the same value of* λ. That is, the polynomial equation (5.14) determines the eigenvalues of **A** uniquely, regardless of the chosen representation. We conclude that

> *the eigenvalues of an operator are a property of the operator itself and do not depend on the choice of basis of the space Ω.*

If we can find *n* linearly independent eigenvectors $\{\vec{x}_k\}$ of **A**, belonging to the corresponding eigenvalues λ_k (not necessarily all different) we can use these vectors to define a basis of Ω. The matrix representation of **A** in this basis is given by (5.1): $\mathbf{A}\vec{x}_j = \vec{x}_i\,A_{ij}$. On the other hand, if $\lambda_j \equiv \lambda'$, then $\mathbf{A}\vec{x}_j = \lambda'\vec{x}_j = \lambda'\delta_{ij}\,\vec{x}_i$. Therefore, since the $\vec{x}_k$ are linearly independent, we must have $A_{ij}=\lambda'\delta_{ij}$. We conclude that, in the eigenvector basis the matrix representation of the operator **A** has the *diagonal* form

$$A = \mathrm{diag}\,(\lambda_1\,,\,\lambda_2\,,\,\ldots\,,\,\lambda_n)\ .$$

Moreover, by the above formula and by the fact that the quantities tr$A$, det$A$ and λ_k are basis-independent (i.e., invariant under basis transformations) it follows that, in *any* basis of Ω,

$$\mathrm{tr}\, A = \lambda_1 + \lambda_2 + ... + \lambda_n \quad , \qquad \det A = \lambda_1 \lambda_2 ... \lambda_n \tag{5.15}$$

Proposition 5.1: Let **A** and **B** be two linear operator on Ω. We assume that **A** and **B** have a common set of n linearly independent eigenvectors $\{\vec{x}_k\}$. Then the operators **A** and **B** *commute*:

$$\mathbf{AB} = \mathbf{BA} \quad \Leftrightarrow \quad [\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA} = 0$$

where [**A**, **B**] denotes the *commutator* of **A** and **B**.

Proof: Since the n vectors $\{\vec{x}_k\}$ are linearly independent, they define a basis of Ω. By assumption, for each value of k the vector $\vec{x}_k$ is an eigenvector of both **A** and **B**, with corresponding eigenvalues, say, α and β. Then,

$$(\mathbf{AB})\,\vec{x}_k \equiv \mathbf{A}\,(\mathbf{B}\,\vec{x}_k) = \mathbf{A}\,(\beta \vec{x}_k) = \beta\,(\mathbf{A}\,\vec{x}_k) = \beta\alpha\,\vec{x}_k$$

and similarly, $(\mathbf{BA})\,\vec{x}_k = \alpha\beta\vec{x}_k$. Thus,

$$(\mathbf{AB})\,\vec{x}_k = (\mathbf{BA})\,\vec{x}_k \quad \Leftrightarrow \quad [\mathbf{A}, \mathbf{B}]\,\vec{x}_k = 0\,,$$

for all k=1,...,n. Now, let $\vec{\Psi} = \xi_i\,\vec{x}_i$ be an arbitrary vector in Ω. Then,

$$[\mathbf{A}, \mathbf{B}]\,\vec{\Psi} = [\mathbf{A}, \mathbf{B}](\xi_i\,\vec{x}_i) = \xi_i\,[\mathbf{A}, \mathbf{B}]\,\vec{x}_i = 0\,, \quad \forall\,\vec{\Psi} \in \Omega\,.$$

This means that [**A**, **B**]=0.

Definition: An operator **A** is said to be *nonsingular* if det$A$≠0 (note that this is a *basis-independent* property). A nonsingular operator is *invertible*, in the sense that an inverse linear operator $\mathbf{A}^{-1}$ on Ω exists such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}_{\mathrm{op}}$, where $\mathbf{1}_{\mathrm{op}}$ is the unit operator. This allows us to write

$$\vec{y} = \mathbf{A}\,\vec{x} \quad \Leftrightarrow \quad \vec{x} = \mathbf{A}^{-1}\,\vec{y}\,.$$

By (5.4) it follows that, if A is the matrix representation of the nonsingular operator **A** in some basis, then *the matrix of the inverse operator* $\mathbf{A}^{-1}$ *is the inverse* A^{-1} *of* A. As is well known, the matrix A may have an inverse iff det$A$≠0, whence the definition of a nonsingular operator. In view of the second relation in (5.15),

all eigenvalues of a nonsingular operator are nonzero.

Indeed, if even one eigenvalue vanishes, then det$A$=0 in *any* representation.

6. Comments

Both the active and the passive view are of importance in Physics. Let us see some examples:

1. The *Galilean transformation* of Classical Mechanics and the *Lorentz transformation* of Relativity[2] are *passive* transformations connecting different inertial frames of reference. When expressed in terms of mathematical equations, all physical laws are required to be invariant in form upon passing from one inertial frame to another.

2. The operators of Quantum Mechanics[3] are *active* transformations from a quantum state to a new state. On the other hand, both states and operators may be represented by matrices in different bases, the transformation from one basis to another being a *passive* transformation. Typically, the basis vectors of the quantum-mechanical space are chosen to be eigenvectors of linear operators representing physical quantities such as energy, angular momentum, etc. In such a basis the related operator is represented by a *diagonal* matrix, the diagonal elements being the *eigenvalues* of the operator. Physically, these eigenvalues give the possible values that a measurement of the associated physical quantity may yield in an experiment.

[2] H. Goldstein, *Classical Mechanics*, 2nd Ed. (Addison-Wesley, 1980).
[3] E. Merzbacher, *Quantum Mechanics*, 3rd Ed. (Wiley, 1998).

Vectors and pseudovectors in electromagnetism

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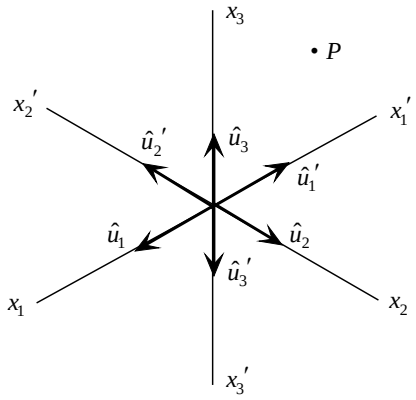
The concept of pseudovectors is simply explained. Application is made to the Maxwell equations of electromagnetism, including the case where hypothetical magnetic charges and currents are present.

1. True vectors and pseudovectors

Perhaps the simplest way to distinguish *vectors* from *pseudovectors* is to examine the way each type of object transforms under *space inversion*.



Let (x_1, x_2, x_3) be an orthogonal system of coordinates, with corresponding unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$. This coordinate system is said to be *right-handed*, since

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3 \ , \quad \hat{u}_2 \times \hat{u}_3 = \hat{u}_1 \ , \quad \hat{u}_3 \times \hat{u}_1 = \hat{u}_2 \tag{1}$$

where the vector (cross) product is defined by the usual *right-hand-rule* convention.

Imagine now that we invert the directions of all three axes, thus obtaining a new coordinate system (x_1', x_2', x_3') with corresponding unit vectors

$$\hat{u}_i' = -\hat{u}_i \quad (i = 1, 2, 3) \tag{2}$$

If we insist on using the right-hand convention, then

$$\hat{u}_1' \times \hat{u}_2' = (-\hat{u}_1) \times (-\hat{u}_2) = \hat{u}_1 \times \hat{u}_2 = \hat{u}_3 = -\hat{u}_3' \quad \text{(etc)}.$$

If, however, we employ the *left-hand* convention, then

$$\hat{u}_1' \times \hat{u}_2' = \hat{u}_3' \ , \quad \hat{u}_2' \times \hat{u}_3' = \hat{u}_1' \ , \quad \hat{u}_3' \times \hat{u}_1' = \hat{u}_2' \tag{3}$$

We say that the system (x_i') is *left-handed*.[1] Thus,

- *the inversion of a right-handed coordinate system is a left-handed system.*

Let P be a point in space. The position of P does not depend, of course, on whether we choose a right-handed or a left-handed system to specify it. However, the *coordinates* of P *do* depend on this choice. Write:

$$P \equiv (x_1, x_2, x_3) \equiv (x_1', x_2', x_3') \ .$$

The two systems of coordinates are related by the set of equations

$$x_i' = -x_i \quad (i=1,2,3) \tag{4}$$

Now, consider a physical object that is described by a vector (e.g., velocity, force, electric or magnetic field, etc.). Assume that in the system (x_i) it is mathematically represented by

$$\vec{A} = A_1\hat{u}_1 + A_2\hat{u}_2 + A_3\hat{u}_3 \equiv \sum_i A_i\hat{u}_i \tag{5}$$

while in the system (x_i') it is represented by

$$\vec{A}' = A_1'\hat{u}_1' + A_2'\hat{u}_2' + A_3'\hat{u}_3' \equiv \sum_i A_i'\hat{u}_i' \tag{6}$$

A *(true) vector* is a geometrical object independent of whether the coordinate system we use is right-handed or left-handed (that is, independent of the "handedness" of the underlying coordinate system). Hence,

$$\vec{A}' = \vec{A} \tag{7}$$

In view of (2), (5), (6) and (7), the components of a vector transform under space inversion according to the relations

$$A_i' = -A_i \quad (i=1,2,3) \tag{8}$$

A *pseudovector* (or *axial vector*), on the other hand, transforms differently:

$$\vec{A}' = -\vec{A} \tag{9}$$

so that, by (2), (5), (6) and (9), its components transform as follows under space inversion:

[1] Note that if only *two* axes of a right-handed coordinate system are inverted, the resulting system is still *right*-handed; if only *one* axis is inverted, the system is *left*-handed.

$$A_i' = A_i \quad (i=1,2,3) \tag{10}$$

Obviously, a pseudovector is *not* an invariant geometrical object since it is dependent upon the handedness of the coordinate system.

Example 1. Let each of $\vec{A}$ and $\vec{B}$ be a vector or a pseudovector. Define the vector (cross) product of these objects in the coordinate systems (x_i) and (x_i') as follows:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad , \quad \vec{A}'\times\vec{B}' = \begin{vmatrix} \hat{u}_1' & \hat{u}_2' & \hat{u}_3' \\ A_1' & A_2' & A_3' \\ B_1' & B_2' & B_3' \end{vmatrix} \tag{11}$$

By taking into account relations (2) and (7) – (10), we conclude the following:

- If *both* $\vec{A}$ and $\vec{B}$ are *vectors* or *both* are *pseudovectors*, then $\vec{A}'\times\vec{B}' = -\vec{A}\times\vec{B}$ so that the cross product is a *pseudovector*.
- If *either* $\vec{A}$ *or* $\vec{B}$ is a *vector*, the other being a *pseudovector*, $\vec{A}'\times\vec{B}' = \vec{A}\times\vec{B}$ so that the cross product is a *vector*.

Example 2. Consider the *del* operator, expressed in the coordinate systems (x_i) and (x_i') as follows:

$$\vec{\nabla} = \sum_{i=1}^{3} \hat{u}_i \frac{\partial}{\partial x_i} \quad , \quad \vec{\nabla}' = \sum_{i=1}^{3} \hat{u}_i' \frac{\partial}{\partial x_i'} \tag{12}$$

We notice that

$$\vec{\nabla}' = \sum_{i=1}^{3} (-\hat{u}_i) \frac{\partial}{\partial(-x_i)} = \vec{\nabla} .$$

Thus, according to (7), the del operator is a (true) *vector* operator. Then, according to Example 1,

- if $\vec{A}$ is a *vector*, its *rot* $\vec{\nabla}\times\vec{A}$ is a *pseudovector*, while
- if $\vec{B}$ is a *pseudovector*, its *rot* $\vec{\nabla}\times\vec{B}$ is a *vector*.

Definition. A quantity Φ is a (true) *scalar* if its value remains invariant under space inversion:

$$\Phi' = \Phi \tag{13}$$

A quantity Φ is a *pseudoscalar* if it changes sign under space inversion:

$$\Phi' = -\Phi \tag{14}$$

Example 3. Let each of $\vec{A}$ and $\vec{B}$ be a vector or a pseudovector. Define the scalar (dot) product of these objects in the coordinate systems (x_i) and (x_i') as follows:

$$\vec{A}\cdot\vec{B}=\sum_{i=1}^{3}A_iB_i\ ,\qquad \vec{A}'\cdot\vec{B}'=\sum_{i=1}^{3}A_i'B_i' \tag{15}$$

We observe the following:

- If *both* $\vec{A}$ and $\vec{B}$ are *vectors* or *both* are *pseudovectors*, then $\vec{A}'\cdot\vec{B}'=\vec{A}\cdot\vec{B}$ so that the dot product is a *scalar*.
- If *either* $\vec{A}$ *or* $\vec{B}$ is a *vector*, the other being a *pseudovector*, $\vec{A}'\cdot\vec{B}'=-\vec{A}\cdot\vec{B}$ so that the dot product is a *pseudoscalar*.

Example 4. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ be (true) vectors. Then $\vec{B}\times\vec{C}$ is a pseudovector, so that $\vec{A}\cdot(\vec{B}\times\vec{C})$ is a pseudoscalar.

Example 5. Regarding the divergence of a vector quantity, we have the following:

- If $\vec{A}$ is a (true) *vector*, its *div* $\vec{\nabla}\cdot\vec{A}$ is a (true) *scalar*, while
- if $\vec{B}$ is a *pseudovector*, its *div* $\vec{\nabla}\cdot\vec{B}$ is a *pseudoscalar*.

Example 6. The Laplace operator

$$\nabla^2\equiv\vec{\nabla}\cdot\vec{\nabla}=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^{\,2}} \tag{16}$$

is a scalar operator. Thus, if Φ is either a scalar or a pseudoscalar function, transforming under space inversion according to the general rule

$$\Phi'(x_1', x_2', x_3') = \pm\,\Phi(x_1, x_2, x_3) \tag{17}$$

(where the plus sign corresponds to a scalar while the minus sign to a pseudoscalar), then $\nabla^2\Phi$ is a scalar or a pseudoscalar function, respectively. Note also that

- *the grad* $\vec{\nabla}\Phi$ *of a scalar (pseudoscalar) function is a vector (pseudovector) function.*

2. Applications in electromagnetism

By its definition, $\vec{E} = \vec{F}_e / q$, and by the fact that the electric force $\vec{F}_e$ is a (true) vector,[2] we see that

- *the electric field is a vector.*

On the other hand, since both the magnetic force $\vec{F}_m = q(\vec{v} \times \vec{B})$ and the velocity $\vec{v}$ of a charged particle are vectors, we conclude that

- *the magnetic field is a pseudovector*

(cf. Example 1 in Sec. 1).

Consider the Maxwell equations:

$$\begin{aligned} &(a) \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\ &(b) \quad \vec{\nabla} \cdot \vec{B} = 0 \\ &(c) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &(d) \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{18}$$

Equation (18*a*) is consistent with the fact that the electric field is a vector and the charge density ρ is a scalar function. In (18*c*) the electric field is a vector, thus its *rot* on the left-hand side is a pseudovector (cf. Example 2 in Sec. 1); this is consistent with the fact that the magnetic field is a pseudovector. In (18*d*) the magnetic field is a pseudovector, thus its *rot* on the left-hand side is a vector; this is consistent with the fact that both the electric field and the current density are vectors.

Consider the Poynting vector

$$\vec{N} = \vec{E} \times \vec{H} = \frac{1}{\mu} (\vec{E} \times \vec{B}) \ .$$

Since the electric field is a vector while the magnetic field is a pseudovector, their cross product on the right-hand side must be a vector; therefore so is the Poynting vector on the left. This was to be expected, since the direction of flow of electromagnetic energy is independent of whether our coordinate system is right-handed or left-handed.

[2] In general, a force is a physically measurable quantity that cannot depend on the handedness of our coordinate system.

3. The inclusion of magnetic charges and currents

Although magnetic charges and magnetic currents have not been observed so far in Nature, their existence cannot be precluded in principle. If such quantities are assumed to exist, the Maxwell equations must be generalized accordingly, as follows (the index *e* stands for "*electric*" while the index *m* stands for "*magnetic*"):

$$
\begin{aligned}
&(a) \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\varepsilon_0} \\
&(b) \quad \vec{\nabla}\cdot\vec{B} = \mu_0\,\rho_m \\
&(c) \quad \vec{\nabla}\times\vec{E} = -\mu_0\,\vec{J}_m - \frac{\partial\vec{B}}{\partial t} \\
&(d) \quad \vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}_e + \varepsilon_0\mu_0\,\frac{\partial\vec{E}}{\partial t}
\end{aligned}
\tag{19}
$$

As discussed previously, $\vec{E}$ is a vector while $\vec{B}$ is a pseudovector. Moreover, the electric charge density ρ_e is a scalar function while the electric current density $\vec{J}_e$ is a vector function. Since the *div* of the magnetic field is a pseudoscalar, it follows from (19*b*) that

- *the magnetic charge density ρ_m is a pseudoscalar.*

Also, since the *rot* of the electric field is a pseudovector, it follows from (19*c*) that

- *the magnetic current density $\vec{J}_m$ is a pseudovector.*

By taking the *div* of (19*d*) and (19*c*) and by using (19*a*) and (19*b*), respectively, we find two equations of continuity:

$$\vec{\nabla}\cdot\vec{J}_e + \frac{\partial\rho_e}{\partial t} = 0 \tag{20}$$

$$\vec{\nabla}\cdot\vec{J}_m + \frac{\partial\rho_m}{\partial t} = 0 \tag{21}$$

The physical meaning of these relations is that the electric and the magnetic charge are separately conserved. Notice that (20) is a scalar equation while (21) is a pseudoscalar equation [the *div* of a vector (pseudovector) is a scalar (pseudoscalar)]. On the other hand, by taking the *rot* of (19*c*) and (19*d*) and by using the vector identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

together with the Maxwell system (19), we derive separate non-homogeneous wave equations for the electric and the magnetic field:

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0}\vec{\nabla}\rho_e + \mu_0\left(\vec{\nabla}\times\vec{J}_m + \frac{\partial\vec{J}_e}{\partial t}\right) \tag{22}$$

$$\nabla^2\vec{B} - \varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\left(\vec{\nabla}\rho_m - \vec{\nabla}\times\vec{J}_e + \varepsilon_0\mu_0\frac{\partial\vec{J}_m}{\partial t}\right) \tag{23}$$

Notice that (22) is a vector equation while (23) is a pseudovector equation [recall that the *rot* of a vector (pseudovector) function is a pseudovector (vector) function].

Technically, the two wave equations (22) and (23), together with the two continuity equations (20) and (21), constitute consistency conditions for the Maxwell system (19). This system may be regarded as a sort of Bäcklund transformation relating fields and sources.[3]

[3] See https://arxiv.org/abs/1901.08058 and http://metapublishing.org/index.php/MP/catalog/book/62.

Motion of a charged particle in a uniform magnetic field

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Several aspects of the motion of a charged particle in a uniform magnetic field are examined, both by physical arguments and by explicit solution of the differential equation of motion.

Problem

A particle of mass m and charge $q>0$ enters a uniform magnetic field $\vec{B}$ with initial velocity $\vec{v}_0$ perpendicular to the field. The magnetic field is assumed to be oriented in the positive z-direction.

1. Show that the particle will execute uniform circular motion on the xy-plane and determine the radius r of this motion.

2. Show that the larger the momentum of the particle, the smaller the curvature of the path. Interpret this physically.

3. Determine the angular velocity ω of the particle and show that the period of revolution is independent of the size of the orbit.

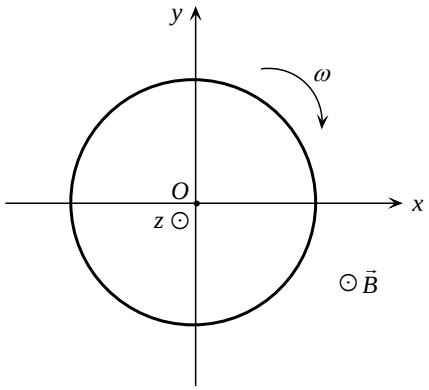
4. Suppose that the magnitude B of the magnetic field increases with time, although the field remains uniform (i.e., *spatially* constant) at all times. Show that the increase of B produces a decrease of the size of the orbit.

5. Assume now that the particle enters the magnetic field in a direction that is *not* perpendicular to the field. Show that the motion of the particle will be uniform, while the projection of this motion onto the xy-plane will be uniform circular with angular velocity ω equal to that found in part 3. Describe the path geometrically.

6. Show that the radiation losses due to acceleration become more significant the smaller the mass of the particle.

7. By solving the differential equation of motion of the charged particle, derive explicit expressions for the coordinates (x, y, z) of the particle as functions of time t. Demonstrate that the projection of the motion onto the xy-plane is uniform circular, as found previously, and verify the expression for the angular velocity ω. Explain why this planar motion is clockwise for the given direction of $\vec{B}$.

Solution



Both the *z*-axis and the magnetic field are normal to the page and directed toward the reader; the direction of motion is clockwise (why?).

1. The charged particle is subject to a magnetic force

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{1}$$

where, in components, $\vec{v} = v_x \hat{u}_x + v_y \hat{u}_y + v_z \hat{u}_z$ and $\vec{B} = B\hat{u}_z$ $(B = |\vec{B}|)$. Then,

$$\vec{F} = qB(v_y \hat{u}_x - v_x \hat{u}_y) \tag{2}$$

which is a vector in the *xy*-plane; the same is true, therefore, with regard to the acceleration of the particle (assuming no other forces act on it). Given that, by assumption, the initial velocity also is a vector in the *xy*-plane, we conclude that the motion of the particle takes place on that plane.

As seen in (1), the total force on the particle is normal to the particle's velocity, i.e., normal to the trajectory of the particle. This means that the particle moves at *constant speed* inside the magnetic field (see, e.g., Section 2.4 of [1] and Sec. 7.1 of [2]). In other words, the particle executes *uniform* curvilinear motion. We must now show that this motion is *circular*. Indeed, the magnitude of the magnetic force is

$$F = qvB = constant \tag{3}$$

where v is the (constant) speed of the particle, equal to the initial speed v_0, and where we have taken into account that the velocity vector is always perpendicular to the magnetic field. Now, since the motion is uniform, the total force (1) is purely centripetal. Hence, $F=mv^2/\rho$, where ρ is the radius of curvature at any point of the trajectory (see Sec. 3.6 of [1]). Given that both v and F are constant, it follows that ρ is constant also; that is, the motion is circular. We may place the center of the circle at the origin O of our coordinate system (in particular, of the *xy*-plane) so that the radius ρ of the circle equals the distance r of the particle from O. From $F=mv^2/r$, and by using (3), we find:

$$r = \frac{mv}{qB} \tag{4}$$

2. Let $p=mv$ be the (constant) magnitude of the momentum of the particle. Relation (4) may then be rewritten as $r=p/qB$. We observe that r is an increasing function of p: the larger the momentum, the larger the radius and, therefore, the smaller the curvature of the path. Physically, this means that as the momentum increases it becomes more difficult for the magnetic field to produce a change in the direction of motion of the particle.

3. We write $v=\omega r$, where ω is the angular velocity. Substituting this into (4), we find

$$\omega = \frac{qB}{m} \tag{5}$$

We notice that ω is independent of the radius r of the orbit; so is, therefore, the period $T=2\pi/\omega$ of the circular motion.

4. Since $v=v_0$=constant, independent of the magnetic-field strength B, a change of B will not affect the speed of the particle. From (4) it then follows that an increase of B will produce a decrease of r, i.e., of the size of the orbit. This means that the particle will revolve closer to the z-axis. This effect is used in fusion reactors to achieve plasma heating and confinement.

5. As argued in part 1 of the problem, since the total force on the particle is normal to the particle's velocity, the speed v of the particle is constant, equal to the initial speed v_0, and the motion is *uniform curvilinear*. Furthermore, the total force, given by (1) and (2), is a vector parallel to the xy-plane, and so is the acceleration of the particle. These results are independent of the direction of the initial velocity of the charge upon its entrance into the magnetic field. Notice also that Eq. (2) is valid even if the velocity has a z-component.

The motion, however, is no longer expected to be planar if the direction of the velocity has a z-component, as will now be assumed to be the case. Let us write

$$\vec{v} = \vec{v}' + v_z \hat{u}_z \quad \text{where} \quad \vec{v}' = v_x \hat{u}_x + v_y \hat{u}_y \equiv \text{vector parallel to the } xy\text{-plane} \tag{6}$$

Since the z-component of the acceleration is zero, the velocity does not change in the z-direction; that is, $v_z=v_{0z}=$*constant*. Hence, along the z-axis (which is parallel to the magnetic field) the motion is uniform rectilinear. Regarding the motion parallel to the xy-plane, we note the following:

$$\vec{F} \cdot \vec{v}' = \vec{F} \cdot \left(\vec{v} - v_z \hat{u}_z \right) = 0 ,$$

since by (1) the total force is normal to the velocity, while by (2) the force is also normal to the z-axis. Alternatively, by using (2) and (6) we have:

$$\vec{F}\cdot\vec{v}' = qB(v_y\hat{u}_x - v_x\hat{u}_y)\cdot(v_x\hat{u}_x + v_y\hat{u}_y) = qB(v_yv_x - v_xv_y) = 0\,.$$

It follows that the motion parallel to the *xy*-plane is uniform curvilinear, with speed equal to

$$v' = \left(v^2 - {v_z}^2\right)^{1/2} = \left({v_0}^2 - {v_{0z}}^2\right)^{1/2} \equiv \text{constant}$$

where we have used the facts that $v=v_0$ and $v_z=v_{0z}$. Furthermore,

$$\vec{v}\times\vec{B} = (\vec{v}' + v_z\hat{u}_z)\times(B\hat{u}_z) = \vec{v}'\times\vec{B}\ ,$$

so that, by (1),

$$F = q\,|\vec{v}'\times\vec{B}\,| = qv'B \equiv \text{constant} \tag{7}$$

If ρ' is the radius of curvature of the projection of the trajectory onto the *xy*-plane, then, given that *F* is purely centripetal, we have:

$$F = m\frac{v'^2}{\rho'} \ \Rightarrow\ \rho' = m\frac{v'^2}{F} \equiv \text{constant} \tag{8}$$

(since both v' and *F* are constant). This means that the projection of the motion onto the *xy*-plane is uniform circular. Overall, the motion of the charge is the resultant of a uniform rectilinear motion parallel to the magnetic field, and a uniform circular motion on a plane perpendicular to the field. The trajectory is a *helix* (*uniform helical motion*). By (7) and (8) we get the radius of the circular projection of the motion:

$$\rho' = \frac{mv'}{qB} \tag{9}$$

Then, by writing $v'=\omega\rho'$, we find that the angular velocity ω is again given by (5); that is, $\omega=qB/m$.

6. The total power radiated by a slowly moving accelerating charge is given by *Larmor's formula* (see Sec. 10.12 of [2])

$$P = \frac{q^2a^2}{6\pi\varepsilon_0c^3} \tag{10}$$

where *a* is the magnitude of the acceleration. Assuming that the charged particle is moving circularly on a plane normal to the magnetic field, and taking Eq. (3) into account, we have: $a=F/m=qvB/m$, where *v* is the constant speed of the particle. We observe that, for given values of *q*, *v* and *B*, the smaller the mass *m* of the particle, the greater the radiated power *P* and hence the greater the power losses. That is, radiation losses become increasingly significant as the mass of the particle decreases. Thus, for example, protons radiate far less than electrons in a cyclical accelerator.

7. The equation of motion of the charged particle is

$$m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})\,.$$

By expanding the left-hand side into components, by using Eq. (2) for the right-hand side, and by equating corresponding components on the two sides of the equation, we obtain the following system of differential equations:

$$\frac{dv_x}{dt} = \omega v_y\ , \quad \frac{dv_y}{dt} = -\omega v_x\ , \quad \frac{dv_z}{dt} = 0 \tag{11}$$

where we have put $\omega = qB/m$. Notice that the expression for ω is the same as that found previously for the angular velocity of the circular projection of the motion on the *xy*-plane.

The system (11) may be integrated by employing the methods described in [3] (cf., in particular, Sec. 4.1 and 5.1). The solution of the system is

$$v_x = A\cos(\omega t - \alpha)\,, \quad v_y = -A\sin(\omega t - \alpha)\,, \quad v_z = \lambda \tag{12}$$

where the $A>0$, α, λ are arbitrary constants. We notice that the speed of the particle is constant, equal to $v=(A^2+\lambda^2)^{1/2}$; the motion is thus uniform. The constants A, α, λ can be expressed in terms of the components (v_{0x} , v_{0y} , v_{0z}) of the initial velocity. Setting $t=0$ in (12), we find:

$$A = (v_{0x}^{\,2} + v_{0y}^{\,2})^{1/2}\ , \quad \lambda = v_{0z}\ , \quad \alpha = \arctan(v_{0y}/v_{0x})\,.$$

Relations (12) are rewritten as a system of differential equations:

$$dx/dt = A\cos(\omega t - \alpha)\,, \quad dy/dt = -A\sin(\omega t - \alpha)\,, \quad dz/dt = \lambda$$

the solution of which system is (by ignoring arbitrary constants)

$$x = (A/\omega)\sin(\omega t - \alpha)\,, \quad y = (A/\omega)\cos(\omega t - \alpha)\,, \quad z = \lambda t \tag{13}$$

Equations (13) express the coordinates of the particle as functions of time.

Projected to the *xy*-plane, the motion of the particle is uniform circular of radius $r=A/\omega$ and with angular velocity $\omega = qB/m$. Define now the function $\theta(t)$ by

$$\theta(t) \equiv \alpha - \omega t + \pi/2 \quad \Leftrightarrow \quad \omega t - \alpha = \pi/2 - \theta(t)\,.$$

Equations (13) are then rewritten as

$$x = r\cos\theta(t)\ , \quad y = r\sin\theta(t)\,.$$

We observe that the pair (r, θ) represents polar coordinates on the *xy*-plane, describing the circle $r=A/\omega=const.$ We also notice that, by its definition, $\theta(t)$ is a *decreasing* function of t ; that is, the polar angle θ decreases with time. This suggests that the circular projection of the path on the *xy*-plane is traversed in the *negative* direction, i.e., *clockwise*.

It also follows from (13) that the motion in the *z*-direction is uniform rectilinear. The overall path of the particle is a *helix* and the motion is, therefore, *uniform helical*.

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C. J. Papachristou,

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http://metapublishing.org/index.php/MP/catalog/book/52 ;
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[3] *Aspects of Integrability of Differential Systems and Fields* (Springer, 2019),
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Center of mass of a system of particles

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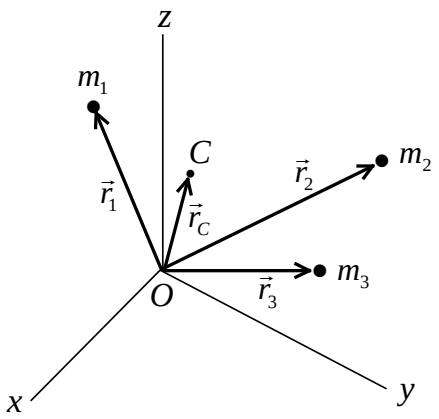
1. Definition of the center of mass

Consider a system of particles of masses m_1 , m_2 , m_3 ,... Assume that at some particular moment the particles are located at the points of space with corresponding position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$,$\cdots$, relative to a reference point O which is typically chosen to be the origin of an inertial[1] frame of reference (see figure).



The total mass of the system is

$$M = m_1 + m_2 + m_3 + \cdots = \sum_i m_i \tag{1}$$

The *center of mass* of the system is defined as the point C of space having the position vector

$$\vec{r}_C = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots) = \frac{1}{M}\sum_i m_i\vec{r}_i \tag{2}$$

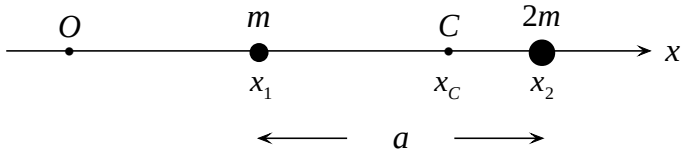
In relation (2) the position vectors of the particles and of the center of mass are defined with respect to the fixed origin O of our coordinate system. If we choose a different reference point O', these position vectors will, of course, change. However, as will be shown below, the position of the center of mass C *relative to the system of particles* will remain the same, regardless of the choice of reference point.

[1] At least, insofar as Newton's laws are to be used.

If (x_i , y_i , z_i) and (x_C , y_C , z_C) are the coordinates of m_i and C, respectively, we can replace the vector relation (2) with three scalar equations:

$$x_C = \frac{1}{M}\sum_i m_i x_i \ , \quad y_C = \frac{1}{M}\sum_i m_i y_i \ , \quad z_C = \frac{1}{M}\sum_i m_i z_i \tag{3}$$

As an example, consider two particles of masses $m_1=m$ and $m_2=2m$, located at points x_1 and x_2 of the x-axis. Call $a = x_2 - x_1$ the distance between these particles:



The total mass of the system is $M=m_1+m_2=3m$. From relations (3) it follows that the center of mass C of the system is located on the x-axis. Indeed, $y_i=z_i=0$ (i=1,2) so that $y_C=z_C=0$ (the y and z-axes have not been drawn). Furthermore,

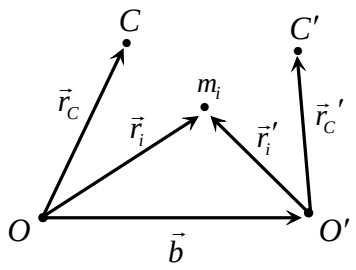
$$x_C = \frac{1}{M}(m_1x_1 + m_2x_2) = \frac{1}{3}(x_1 + 2x_2) = x_1 + \frac{2}{3}a$$

where we have used the fact that $x_2 = x_1 + a$. Thus, the center of mass C is located at a distance $2a/3$ from m. Note that the position of C *relative to the system of particles* does not depend on the choice of the reference point O with respect to which the coordinates of the particles are determined.

As the above example demonstrates, the position of the center of mass does not necessarily coincide with the position of a particle of the system. (Give examples of systems in which a particle is located at C, as well as of systems where no such coincidence occurs.)

2. Independence from the point of reference

We must now show that the location of C in space does not depend on the choice of the reference point O. Let us assume for the moment, however, that the position of C *does* depend on the choice of reference point. So, let C and C' be two different positions of the center of mass, corresponding to the reference points O and O'. We call $\vec{r}_C$ and $\vec{r}_C{}'$ the position vectors of C and C' with respect to O and O', respectively, and we let $\vec{r}_i$ and $\vec{r}_i{}'$ be the position vectors of the particle m_i relative to O and O'. For convenience, we denote by $\vec{b}$ the vector $\overrightarrow{OO'}$ (see figure).



The defining equation (2), expressed successively for O and O', yields

$$\vec{r}_C = \frac{1}{M}\sum_i m_i \vec{r}_i \ , \quad \vec{r}_C{}' = \frac{1}{M}\sum_i m_i \vec{r}_i'$$

where $\vec{r}_i' = \vec{r}_i - \vec{b}$. Now,

$$\overrightarrow{CC'} = \overrightarrow{CO} + \overrightarrow{OO'} + \overrightarrow{O'C'} = -\vec{r}_C + \vec{b} + \vec{r}_C{}' \quad \Rightarrow$$
$$\overrightarrow{CC'} = -\frac{1}{M}\sum_i m_i \vec{r}_i + \vec{b} + \frac{1}{M}\sum_i m_i \vec{r}_i' = \vec{b} - \frac{1}{M}\sum_i m_i (\vec{r}_i - \vec{r}_i')$$
$$= \vec{b} - \frac{1}{M}\sum_i m_i \vec{b} = \vec{b} - \frac{1}{M}\left(\sum_i m_i\right)\vec{b} = \vec{b} - \frac{1}{M} M \vec{b} = 0$$

which suggests that the points C and C' coincide. Hence, the center of mass of the system is a uniquely determined point of space, independent of the origin of our coordinate system.

3. Center of mass and Newton's laws

We define the *total (linear) momentum* of the system at time t, relative to an inertial reference frame, as the vector sum

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \tag{4}$$

Let $\vec{F}_i$ be the *external* force acting on m_i at this instant. The *total external force* acting on the system at time t is $\vec{F}_{\text{ext}} = \sum_i \vec{F}_i$. By Newton's 2nd and 3rd laws we find that

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \tag{5}$$

[see, e.g., Papachristou (2020)]. We now prove the following:

1. *The total momentum of the system is equal to the momentum of a hypothetical particle having mass equal to the total mass M of the system and moving with the velocity of the center of mass of the system.*
2. *The equation of motion of the center of mass of the system is that of a hypothetical particle of mass equal to the total mass M of the system, subject to the total external force* $\vec{F}_{\text{ext}}$ *acting on the system.*

Proof:

1. Differentiating (2) with respect to time, we find the velocity of the center of mass of the system:

$$\vec{v}_C = \frac{d\vec{r}_C}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum_i m_i \vec{r}_i\right) = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} \quad \Rightarrow$$

$$\vec{v}_C = \frac{1}{M}\sum_i m_i \vec{v}_i = \frac{1}{M}\sum_i \vec{p}_i \tag{6}$$

Combining this with (4), we have:

$$\vec{P} = M\,\vec{v}_C \tag{7}$$

2. Differentiating (7), we have:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(M\,\vec{v}_C) = M\,\frac{d\vec{v}_C}{dt} = M\,\vec{a}_C$$

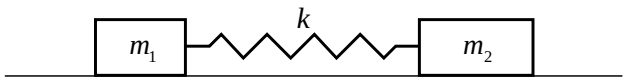
where $\vec{a}_C$ is the acceleration of the center of mass. Hence, by (5),

$$\vec{F}_{\text{ext}} = M\,\vec{a}_C \tag{8}$$

A system of particles is said to be *isolated* if (*a*) it is not subject to any external interactions (a situation that is only theoretically possible) or (*b*) the total external force on the system is zero: $\vec{F}_{\text{ext}} = 0$. In this case, relations (5) and (7) lead to the following conclusions:

1. *The total momentum of an isolated system of particles retains a constant value relative to an inertial frame of reference (principle of conservation of momentum).*
2. *The center of mass C of an isolated system of particles moves with constant velocity relative to an inertial reference frame.*

As an example, consider two masses m_1 and m_2 connected to each other with a spring. The masses can move on a frictionless horizontal plane, as shown in the figure:



The system may be considered isolated since the total external force on it is zero (explain this!). Thus, the total momentum of the system and the velocity of the center of mass *C* remain constant while the two masses move on the plane. Note that the *internal* force $F_{\text{int}}=k\Delta l$, where Δl is the deformation of the spring relative to its natural length, *cannot* produce any change to the total momentum and the velocity of *C*.

4. Center of mass and angular momentum

The *total angular momentum* of the system at time *t*, relative to an arbitrary point *O*, is defined as

$$\vec{L} = \sum_i \vec{L}_i = \sum_i m_i (\vec{r}_i \times \vec{v}_i) \tag{9}$$

In particular, the total angular momentum relative to the center of mass *C* of the system is

$$\vec{L}' = \sum_i m_i (\vec{r}_i' \times \vec{v}_i') \tag{10}$$

where primed quantities are measured with respect to *C*. We have:

$$\vec{r}_i = \vec{r}_i' + \vec{r}_C \quad , \quad \vec{v}_i = \vec{v}_i' + \vec{v}_C \ .$$

Substituting these into (9) and using (1) and (10), we get:

$$\vec{L} = \vec{L}' + M(\vec{r}_C \times \vec{v}_C) + \left[\left(\sum_i m_i \vec{r}_i' \right) \times \vec{v}_C \right] + \left[\vec{r}_C \times \sum_i m_i \vec{v}_i' \right] .$$

But, $\Sigma m_i \vec{r}_i' = 0$ and $\Sigma m_i \vec{v}_i' = 0$, since these quantities are proportional to the position vector and the velocity, respectively, of the center of mass *C* relative to *C* itself. Thus, finally,

$$\vec{L} = \vec{L}' + M(\vec{r}_C \times \vec{v}_C) \tag{11}$$

We may interpret this result as follows:

> *The total angular momentum of the system, with respect to a point O, is the sum of the angular momentum relative to the center of mass ("spin angular momentum") and the angular momentum relative to O, of a hypothetical particle of mass equal to the total mass of the system, moving with the center of mass ("orbital angular momentum").*

Now, suppose *O* is the origin of an *inertial* reference frame. Let $\vec{F}_i$ be the external force acting on m_i at time *t*. The *total external torque* acting on the system at this time, relative to *O*, is given by

$$\vec{T}_{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_i \tag{12}$$

If we make the assumption that all *internal* forces in the system are *central* (as the case is in most physical situations of interest), then the following relation exists between the total angular momentum and the total external torque, both quantities measured relative to *O* [see, e.g., Papachristou (2020)]:

$$\frac{d\vec{L}}{dt} = \vec{T}_{\text{ext}} \tag{13}$$

Equation (13) is strictly valid relative to the origin *O* of an *inertial* frame. If the system of particles is *isolated*, the center of mass *C* moves with constant velocity (relative to *O*) thus is a proper choice of point of reference for the vector relation (13). That is, (13) is valid with respect to the center of mass of an isolated system. But, what if the system of particles is *not* isolated? Then *C* is *accelerating* (relative to *O*) and it would appear that (13) is not valid relative to *C* in this case. This is not so, however:

> *Equation (13) is always valid with respect to the center of mass C, even when C is accelerating (i.e., even if the system of particles is not isolated)!*

Indeed, by differentiating (11) with respect to time and by using (13), (12) and (8), we have:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}'}{dt} + M(\vec{r}_C \times \vec{a}_C) \left(+M(\vec{v}_C \times \vec{v}_C) \text{ which vanishes}\right) \Rightarrow$$

$$\vec{T}_{\text{ext}} \equiv \sum_i \vec{r}_i \times \vec{F}_i = \frac{d\vec{L}'}{dt} + (\vec{r}_C \times \vec{F}_{\text{ext}}) \Rightarrow$$

$$\frac{d\vec{L}'}{dt} = \sum_i \vec{r}_i \times \vec{F}_i - \left(\vec{r}_C \times \sum_i \vec{F}_i\right) = \sum_i (\vec{r}_i - \vec{r}_C) \times \vec{F}_i$$

$$= \sum_i \vec{r}_i' \times \vec{F}_i = \vec{T}_{\text{ext}}'$$

where $\vec{T}_{\text{ext}}'$ is the total external torque relative to the center of mass.

This observation justifies using (13) to analyze, e.g., the motion of a rolling body on an inclined plane by choosing an axis of rotation that passes through the *accelerating* center of mass of the body.

5. Center of mass and kinetic energy

The *total kinetic energy* of the system relative to an external observer *O* is

$$E_k = \sum_i \frac{1}{2} m_i v_i^2 \tag{14}$$

The total kinetic energy with respect to the center of mass *C* is

$$E_k' = \sum_i \frac{1}{2} m_i v_i'^2 \tag{15}$$

(as before, primed quantities are measured with respect to *C*). We have:

$$\vec{v}_i = \vec{v}_i' + \vec{v}_C \;\; \Rightarrow \;\; v_i^2 = \vec{v}_i \cdot \vec{v}_i = v_i'^2 + v_C^2 + 2\vec{v}_i' \cdot \vec{v}_C \;\; .$$

Substituting this into (14) and using (1) and (15), we get:

$$E_k = E_k' + \frac{1}{2} M v_C^2 + \left(\sum_i m_i \vec{v}_i' \right) \cdot \vec{v}_C \;\; .$$

But, as noted previously, the sum in the last term vanishes, being proportional to the velocity of the center of mass *C* relative to *C*. Thus, finally,

$$E_k = E_k' + \frac{1}{2} M v_C^2 \tag{16}$$

This may be interpreted as follows:

> *The total kinetic energy of the system, relative to an observer O, is the sum of the kinetic energy relative to the center of mass and the kinetic energy relative to O, of a hypothetical particle of mass equal to the total mass of the system, moving with the center of mass.*

6. Adding a particle at – or removing a particle from – the center of mass

We now prove the following:

(*a*) Consider a system of *N* particles of masses m_1 , m_2 , ... , m_N . Let *C* be the center of mass of the system. If a new particle, of mass m , is placed at *C*, the center of mass of the enlarged system of (*N*+1) particles will still be at *C*.

(*b*) Consider a system of *N* particles of masses m_1 , m_2 , ... , m_N . It is assumed that the location of one of the particles, say of m_N , coincides with the center of mass *C* of the system. If we now remove this particle from the system, the center of mass of the remaining system of (*N*–1) particles will still be at *C*.

Proof:

(*a*) The total mass of the original system of N particles is $M=m_1+m_2+...+m_N$. The center of mass of this system is located at the point C with position vector

$$\vec{r}_C = \frac{1}{M}\,(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N)$$

relative to some fixed reference point O. For the additional particle, which we name m_{N+1}, we are given that $m_{N+1}=m$ and $\vec{r}_{N+1} = \vec{r}_C$. The total mass of the enlarged system of (N+1) particles m_1, m_2, ... , m_N, m_{N+1} is $M' = M+m$, and the center of mass of this system, relative to O, is located at

$$\vec{r}_C{}' = \frac{1}{M'}\,(m_1\vec{r}_1 + \cdots + m_N\vec{r}_N + m\vec{r}_C)\ .$$

Now, $m_1\vec{r}_1 + \cdots + m_N\vec{r}_N = M\vec{r}_C$, so that

$$\vec{r}_C{}' = \frac{1}{M+m}\,(M\vec{r}_C + m\vec{r}_C) = \vec{r}_C\ .$$

(*b*) Although this statement is obviously a corollary of part (*a*), we will prove this independently. Here we are given that $\vec{r}_N = \vec{r}_C$. Thus,

$$\frac{1}{M}\,(m_1\vec{r}_1 + \cdots + m_N\vec{r}_N) = \vec{r}_N\ .$$

The mass of the reduced system of (N–1) particles m_1, m_2, ... , m_{N-1} is $M' = M–m_N$, while the center of mass of this system is located at

$$\vec{r}_C{}' = \frac{1}{M'}\,(m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1})\ .$$

But, $m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1} + m_N\vec{r}_N = M\vec{r}_N \quad \Rightarrow$

$$m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1} = (M - m_N)\,\vec{r}_N = M'\vec{r}_N\ .$$

Thus, finally,

$$\vec{r}_C{}' = \frac{1}{M'}M'\vec{r}_N = \vec{r}_N = \vec{r}_C\ .$$

7. Center of mass of a continuous mass distribution

A *rigid body* is a physical object the structure of which exhibits a *continuous* mass distribution. Such an object can be considered as a system consisting of an enormous (practically infinite) number of particles of infinitesimal masses dm_i, placed in such a way that the distance between any two neighboring particles is zero. The total mass of the body is

$$M = \sum_i dm_i = \int dm$$

where the sum has been replaced by an integral due to the fact that the dm_i are infinitesimal and the distribution of mass is continuous.

A point in a rigid body can be specified by its position vector $\vec{r}$, or its coordinates (x, y, z), relative to the origin O of some frame of reference. Let dV be an infinitesimal volume centered at $\vec{r} \equiv (x, y, z)$, and let dm be the infinitesimal mass contained in this volume element. The *density* ρ of the body at point $\vec{r}$ is defined by

$$\rho(\vec{r}) = \rho(x, y, z) = \frac{dm}{dV} \ .$$

Then,

$$dm = \rho(\vec{r})\, dV$$

and the total mass of the body is written

$$M = \int \rho(\vec{r})\, dV$$

where the integration takes place over the entire volume of the body. (The integral is in fact a *triple* one since, in Cartesian coordinates, $dV=dxdydz$.) The center of mass C of the body is found by using (2):

$$\vec{r}_C = \frac{1}{M} \sum_i (dm_i)\, \vec{r}_i = \frac{1}{M} \int \vec{r}\, dm \quad \Rightarrow$$

$$\vec{r}_C = \frac{1}{M} \int \vec{r}\, \rho(\vec{r})\, dV \tag{17}$$

where the $\vec{r}$ and $\vec{r}_C$ are measured relative to the origin O of our coordinate system. (Remember, however, that the location of *C with respect to the body* is uniquely determined and is independent of the choice of the reference point O.)

In a *homogeneous* body the density has a constant value ρ, independent of $\vec{r}$. Then,

$$M = \int \rho\, dV = \rho \int dV = \rho V$$

where V is the total volume of the body. Also, from (17) we have:

$$\vec{r}_C = \frac{\rho}{M}\int \vec{r}\, dV = \frac{1}{V}\int \vec{r}\, dV \tag{18}$$

Imagine now that, instead of a mass distribution in space, we have a *linear* distribution of mass (e.g., a very thin rod) along the x-axis. We define the *linear density* of the distribution by

$$\rho(x) = \frac{dm}{dx}\ .$$

The total mass of the distribution is

$$M = \int dm = \int \rho(x)\, dx\ .$$

The position of the center of mass of the distribution is given by

$$x_C = \frac{1}{M}\int x\, dm = \frac{1}{M}\int x\, \rho(x)\, dx \tag{19}$$

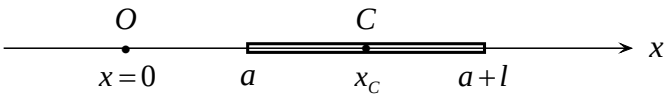
If the density ρ is constant, independent of x, then

$$M = \int \rho\, dx = \rho \int dx = \rho\, l$$

where l is the total length of the distribution. Furthermore,

$$x_C = \frac{\rho}{M}\int x\, dx = \frac{1}{l}\int x\, dx \tag{20}$$

As an example, consider a thin, homogeneous rod of length l, placed along the x-axis from $x=a$ to $x=a+l$, as shown in the figure:



By equation (20),

$$x_C = \frac{1}{l}\int_a^{a+l} x\, dx = \frac{1}{2l}\left[(a+l)^2 - a^2\right] = a + \frac{l}{2}\ .$$

That is, the center of mass C of the rod is located at the center of the rod. Note that the location of C on the rod is uniquely determined, independently of the choice of the origin O of the x-axis (although the value of the coordinate x_C does, of course, depend on this choice).

8. Center of mass and center of gravity

We have seen that the center of mass C of a system of particles moves in space as if it were a particle of mass equal to the total mass M of the system, subject to the total external force acting on the system. The same is true for a rigid body. Let us assume that the only external forces that act on the system (or the rigid body) are those due to gravity. The total external force is then equal to the *total weight* of the system:

$$\vec{w} = \sum_i \vec{w}_i = \sum_i (m_i \vec{g}) = \left(\sum_i m_i \right) \vec{g} \quad \Rightarrow$$

$$\vec{w} = M \vec{g} \quad \text{where} \quad M = \sum_i m_i \ .$$

The acceleration of gravity $\vec{g}$ is constant in a region of space where the gravitational field may be considered uniform.

Note that $\vec{w}$ is a sum of forces that act on separate particles (or elementary masses dm_i in the case of a rigid body) located at various points of space. The question now is whether there exists some specific point of application of the total weight $\vec{w}$ of the system and, in particular, of a rigid body. A reasonable assumption is that this point could be the center of mass C of the body, given that, as mentioned above, the point C behaves as if it concentrates the entire mass M of the body and the total external force acting on it. And, in our case, $\vec{w}$ is indeed the total external force due to gravity.

There is a subtle point here, however: In contrast to a point particle (such as the hypothetical "particle" of mass M moving with the center of mass C) that simply changes its location in space, a rigid body may execute a more complex motion, specifically, a combination of translation and rotation. The *translational* motion of the body under the action of gravity is indeed represented by the motion of the center of mass C, if this point is regarded as a "particle" of mass M on which the total force $\vec{w}$ is applied. For the *rotational* motion of the body, however, it is the *torques* of the external forces, rather than the forces themselves, that are responsible. Where should we place the total force $\vec{w}$ in order that the rotational motion it produces on the body be the same as that caused by the simultaneous action of the elementary gravitational forces $d\vec{w}_i = (dm_i)\vec{g}$? Equivalently, where should we place $\vec{w}$ in order that its torque *with respect to any point O* be equal to the total torque of the $d\vec{w}_i$ with respect to O ?

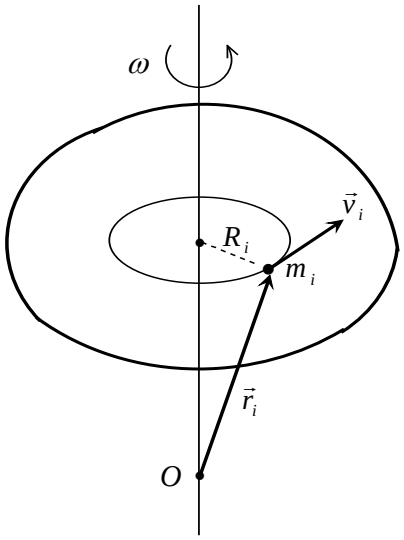
You may have guessed the answer already: at the center of mass C ! [See, e.g., Papachristou (2020).] In conclusion:

> *By placing the total weight $\vec{w}$ of the body at the center of mass C we manage to describe not only the translational but also the rotational motion of the body under the action of gravity.*

It is for this reason that C is frequently called the *center of gravity* of the body. Note that this point does *not necessarily* belong to the body (consider, for example, the cases of a ring and a spherical shell).

9. Mechanical energy of a rigid body

Consider a rigid body rotating with angular velocity ω about an axis passing from a fixed point O of space:



During rotation, every elementary mass m_i in the body moves circularly about the axis of rotation, with the common angular velocity ω. If R_i is the perpendicular distance of m_i from the axis (thus, the radius of the circular path of m_i), the speed of this mass element is $v_i = R_i \omega$. The total *kinetic energy of rotation* is the sum of the kinetic energies of all elementary masses m_i contained in the body:

$$E_{k,rot} = \sum_i \left(\frac{1}{2} m_i v_i^2 \right) = \sum_i \left(\frac{1}{2} m_i R_i^2 \omega^2 \right) = \frac{1}{2} \omega^2 \sum_i m_i R_i^2 \quad \Rightarrow$$

$$E_{k,rot} = \frac{1}{2} I \omega^2 \tag{21}$$

where

$$I = \sum_i m_i R_i^2$$

is the *moment of inertia* of the body relative to the axis of rotation.

Relation (21) represents the total kinetic energy of the body when the latter executes *pure rotation* about a fixed axis. A more general kind of motion is a rotation about an axis that is moving in space. Specifically, assume that the axis of rotation passes from the center of mass C of the body, while C itself moves in space with velocity $\vec{v}_C$. The body thus executes a composite motion consisting of a *translation* of the center of mass C and a *rotation* about C. According to equation (16), the total kinetic energy of the body is the sum of two quantities: a *kinetic energy of translation*,

$$E_{k,trans} = \frac{1}{2} M v_C^2$$

(where M is the mass of the body and v_C is the speed of the center of mass C) and a *kinetic energy of rotation about C*,

$$E_{k,rot} = \frac{1}{2} I_C \omega^2$$

(where ω is the angular velocity of rotation about an axis passing from C, while I_C is the moment of inertia of the body relative to this axis[2]). Hence, the total kinetic energy of the body is

$$E_k = E_{k,trans} + E_{k,rot} = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2 \tag{22}$$

If the body is subject to external forces that are conservative, we can define an *external potential energy* E_p as well as a *total mechanical energy* E, the latter assuming a constant value during the motion of the body:

$$E = E_k + E_p = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2 + E_p = const. \tag{23}$$

For example, if the body moves under the sole action of gravity, its potential energy is

$$E_p = M g y_C$$

where y_C is the vertical distance (the height) of the center of mass C with respect to an arbitrary horizontal plane of reference. Indeed, by relation (3),

$$y_C = \frac{1}{M} \sum_i m_i y_i$$

where y_i is the height of the point of location of the elementary mass m_i in the body. The total gravitational potential energy of the body, equal to the sum of the potential energies of all elementary masses m_i, is then

$$E_p = \sum_i (m_i g y_i) = g \sum_i m_i y_i = M g y_C \ .$$

The total mechanical energy of the body is constant and equal to

$$E = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2 + M g y_C \tag{24}$$

[2] The moment of inertia I relative to an axis parallel to this axis is given by the *parallel-axis theorem* [see, e.g., Papachristou (2020)]. Specifically, $I=I_C+Ma^2$, where a is the perpendicular distance between the two axes.

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[3] http://metapublishing.org/index.php/MP/catalog/book/68

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A note on the principle of superposition in electrodynamics

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In textbooks in electromagnetism the principle of superposition is usually referred to in the context of electrostatics and is justified by Coulomb's law and by the superposition principle for forces postulated in classical mechanics. At a deeper level of analysis, the superposition principle for time-dependent electromagnetic fields is a direct consequence of the linearity of Maxwell's system of equations. The analogous principle for forces is a separate axiom in mechanics, independent of Newton's laws.

In textbooks in electromagnetism, both of intermediate [1-5] and advanced [6-8] level, the principle of superposition is usually referred to in the chapter on electrostatics. The idea is very simple: As experiment shows, the interaction of any two charges is unaffected by the presence of other charges. Thus, by Coulomb's law and by the superposition principle for forces postulated in classical mechanics [9] the electric field created by a system of charges equals the vector sum of the fields due to each charge separately.

Indeed, let $\{q_k\}$ (k=1,2,...) be a set of stationary[1] charges and let $\{\vec{E}_k(\vec{r})\}$ (k=1,2,...) be the corresponding electrostatic fields created separately by each of these charges. We consider a test charge q_0 (not belonging to the set $\{q_k\}$) placed at some point $\vec{r}$ of space and we call $\vec{F}_k$ the force on q_0 due to the field $\vec{E}_k$ created by q_k. By the superposition principle for forces, the total force on q_0 by the electric field of the entire system $\{q_k\}$ is the vector sum $\vec{F} = \sum_i \vec{F}_i$. Consider now a vector field whose value at the location of q_0 is

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q_0} = \sum_i \frac{\vec{F}_i}{q_0} \ .$$

By Coulomb's law, the force on q_0 due to q_i is proportional to q_0, so that the quotient $\vec{F}_i / q_0$ is independent of q_0 and uniquely defines the electric field $\vec{E}_i$ due to q_i at the location of q_0. Hence, the vector sum

$$\vec{E}(\vec{r}) = \sum_i \vec{E}_i(\vec{r})$$

is independent of the test charge q_0 and represents the electric field produced by the entire collection of charges $\{q_k\}$.

[1] Relative to an inertial observer [9].

Notice that the above proof rests critically on two assumptions: (*a*) the force exerted by a charge q_k on q_0 is independent of the forces exerted on q_0 by other charges; (*b*) Coulomb's law is valid. As mentioned above, assumption (*a*) is related to the principle of superposition for forces[2] (one might call it "Newton's fourth law"); namely, the total force on a particle due to its simultaneous interaction with several objects is equal to the vector sum of the forces due to each object acting independently on the particle. As for Coulomb's law, it is the physical content of Gauss' law for the electric field, the latter law constituting the first of Maxwell's equations for the electromagnetic (e/m) field. It is thus an interesting exercise to check that the Maxwell system of equations is consistent with the principle of superposition in its most general form.

The Maxwell equations for the e/m field $(\vec{E},\vec{B})$ is a system of linear first-order partial differential equations:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \qquad & \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} &= 0 \qquad & \vec{\nabla}\times\vec{B} &= \mu_0\,\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where the charge and current densities $\left(\rho(\vec{r},t),\,\vec{J}(\vec{r},t)\right)$ are subject to the equation of continuity

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{2}$$

required for charge conservation.

Consider a region Ω of space and let $\left(\rho_k(\vec{r},t),\,\vec{J}_k(\vec{r},t)\right)$ (k=1,2,...) be a collection of charge and current distributions within Ω. Each pair $(\rho_k,\vec{J}_k)$ is subject to the condition

$$\vec{\nabla}\cdot\vec{J}_k + \frac{\partial\rho_k}{\partial t} = 0 \tag{3}$$

We assume that there are no charges and/or currents in the exterior of Ω, so that the e/m field in Ω is *due exclusively to the sources contained in* Ω. Each individual distribution $(\rho_k,\vec{J}_k)$ will give rise to a corresponding e/m field $(\vec{E}_k,\vec{B}_k)$ satisfying the Maxwell system:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}_k &= \frac{\rho_k}{\varepsilon_0} \qquad & \vec{\nabla}\times\vec{E}_k &= -\frac{\partial\vec{B}_k}{\partial t} \\ \vec{\nabla}\cdot\vec{B}_k &= 0 \qquad & \vec{\nabla}\times\vec{B}_k &= \mu_0\,\vec{J}_k + \varepsilon_0\mu_0\frac{\partial\vec{E}_k}{\partial t} \end{aligned} \tag{4}$$

[2] First stated by Daniel Bernoulli after Newton's death.

We now define a total distribution $(\rho, \vec{J})$ in Ω by

$$\rho(\vec{r},t) = \sum_i \rho_i(\vec{r},t)\ , \quad \vec{J}(\vec{r},t) = \sum_i \vec{J}_i(\vec{r},t) \tag{5}$$

By using (3) and by taking into account the linearity of the *div* and $\partial/\partial t$ operators, the continuity equation (2) may easily be verified for the total distribution (5). We also define the pair of vector functions $(\vec{E}, \vec{B})$ in Ω by

$$\vec{E}(\vec{r},t) = \sum_i \vec{E}_i(\vec{r},t)\ , \quad \vec{B}(\vec{r},t) = \sum_i \vec{B}_i(\vec{r},t) \tag{6}$$

where $(\vec{E}_k, \vec{B}_k)$ is the e/m field produced by the distribution $(\rho_k, \vec{J}_k)$. We propose to show that $(\vec{E}, \vec{B})$ is the e/m field in Ω produced by the total distribution $(\rho, \vec{J})$. For this to be the case it is sufficient that the pair $(\vec{E}, \vec{B})$ satisfy the Maxwell system (1) for the distribution $(\rho, \vec{J})$, given that, by assumption, there are no sources outside Ω that might contribute to the e/m field inside Ω.

By substituting the sums (6) for the vector functions $(\vec{E}, \vec{B})$ into Maxwell's equations (1) and by taking relations (4) and (5) into account, it is not hard to show that the system (1) is indeed satisfied. For example,

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \sum_i \vec{B}_i = \sum_i \vec{\nabla} \times \vec{B}_i \overset{(4)}{=} \sum_i \left(\mu_0 \vec{J}_i + \varepsilon_0 \mu_0 \frac{\partial \vec{E}_i}{\partial t} \right)$$

$$= \mu_0 \sum_i \vec{J}_i + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \sum_i \vec{E}_i = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

We conclude that

> if the distributions $(\rho_k, \vec{J}_k)$ independently produce the corresponding e/m fields $(\vec{E}_k, \vec{B}_k)$ (k=1,2,...) in a region Ω, then the e/m field in Ω produced by the total distribution (5) is given by the vector sums in (6).

Notice that this generalized form of the superposition principle for time-dependent e/m fields rests on the linearity of Maxwell's differential equations. Thus, in electromagnetism the principle of superposition is "built into" the fundamental equations of the theory from the outset, which is not the case with Newtonian mechanics where the analogous principle for forces must be added *a posteriori* to the system of basic laws (see, e.g., [9]).

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The physical meaning of Fermi-Dirac statistics

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In this article we use material from this author's textbook *"Introduction to Electromagnetic Theory and the Physics of Conducting Solids"* (Springer, 2020).[2]

1. Occupation density and density of states

We consider a quantum system consisting of a large number of identical particles. We assume that the energy of each particle may take on certain values $E_1, E_2, E_3, \ldots$, characteristic for this system. We say that each particle may *occupy* one of the available energy levels $E_1, E_2, E_3, \ldots$, of the system. We also assume that the system occupies *unit volume*. Hence, all physical quantities concerning this system will be specified *per unit volume*. At some instant the particles are distributed to the various energy levels so that n_i particles (per unit volume) occupy the level E_i (which means that each of these n_i particles has energy E_i), as seen in Fig. 1.

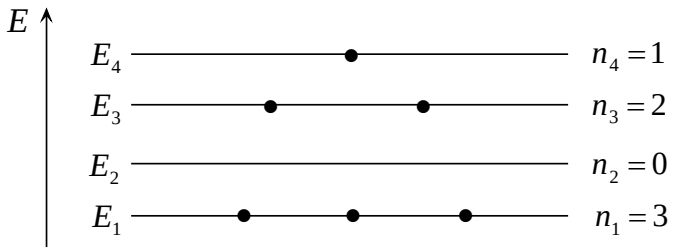

Fig. 1

The total number of particles in the system is equal to

$$n = \sum_i n_i \tag{1}$$

while the total energy of the system is[3]

$$U = \sum_i n_i E_i \tag{2}$$

The ordered set $(n_1, n_2, n_3, \ldots) \equiv (n_i)$ constitutes a *partition* and defines a *microstate* of the system, compatible with the macroscopic state determined by the number n of particles, the total energy U, etc. By the expression (2) we implicitly assume that the particles do not interact (or, at least, do not interact too strongly) with one another, so that we may define an average energy separately for each particle. This is approximately true for the molecules of ideal gases, as well as for the free electrons in metals.

If the system is *isolated* (i.e., does not exchange matter or energy with its surroundings) the n and U are constant. However, Eqs. (1) and (2) do not determine the

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[2] Manuscript: https://arxiv.org/abs/1711.09969

[3] We assume that the values E_i of the energy are characteristic of the specific *kind* of system and do *not* depend on the total number of particles (thus are independent of the volume of the system).

partition (n_i) uniquely, given that different partitions (n_i), (n_i'), (n_i''), etc, may correspond to the same values of n and U. Now, for given n and U there is a *most probable* partition (microstate). When the system is in that state of maximum probability we say that it is in *statistical equilibrium* (in thermodynamics the term *thermal equilibrium* is used). When an isolated system reaches a state of statistical equilibrium it tends to remain in that state – unless, of course, it is disturbed by some external action. Furthermore, in a state of equilibrium the system has a well-defined, constant temperature T. As a rule, *we will always assume that the systems we consider are in statistical equilibrium*.

Assume now that the particles in the system have energies that vary *continuously* from E_1 to E_2 ($E_1 \leq E \leq E_2$) instead of taking on discrete values $E_1, E_2, E_3, ...$ This is the case for the free electrons in a metal – their energies varying continuously within the limits of the conduction band – as well as for the molecules of an ideal gas that occupies a large volume. In this case there is an infinite number of energy levels varying between the limit values E_1 and E_2 . The distribution of the particles of the system among these levels is now described with the aid of a function $n(E)$, to be called the *occupation density*, defined as follows:

> *The product $n(E)dE$ represents the number of particles, per unit volume, whose energies have values between E and $E+dE$.*

One may say that the occupation density $n(E)$ expresses the *distribution of energy* in the system. More accurately, for a given value E of the energy, the corresponding value $n(E)$ describes the "tendency" of the particles in the system to occupy energy levels in the vicinity of E: a larger $n(E)$ means a larger number of particles in the energy region between E and $E+dE$.

The total number n of particles in the system, per unit volume, is equal to

$$n = \int_{E_1}^{E_2} n(E)\, dE \tag{3}$$

In the case of metals, n represents the concentration of free electrons (number of free electrons per unit volume); that is, the *electronic density* of the metal.

The quantum state of a particle in the system is described with the aid of a set of quantum numbers, characteristic of the particular kind of system. In general, to every value E of the energy (that is, to every energy level) there correspond many different quantum states. Some of them will be occupied by particles while others will be vacant. In a manner similar to the definition of the occupation density $n(E)$, we define the *density of states* $N(E)$ as follows:

> *The product $N(E)dE$ represents the number of states, per unit volume of the system, whose energies have values between E and $E+dE$.*

Like the occupation density, the density of states is only defined if the energies of the particles vary in a continuous manner. It is also obvious that we cannot expect to find any particles in an energy region where there are no allowable quantum states. Therefore, $n(E)=0$ when $N(E)=0$. The converse is *not* true, given that there may exist allowable energy regions where all states are vacant (this is, e.g., the case with the upper part of the conduction band of a metal).

2. The ideal gas

An important problem in statistical physics is the distribution of energy in an *ideal monatomic gas*. Since every gas molecule consists of a single atom, its energy is purely *translational kinetic* (there is no intermolecular potential energy, nor is there the rotational or the vibrational kinetic energy typical of a composite molecule). The molecular energy levels are thus given by the relation $E_i = \frac{1}{2} m v_i^2$, where m is the mass of a molecule and where v_i are the possible values of the velocity of the molecules. For given physical conditions, each level E_i is occupied by all molecules having a common speed v_i.

The gas is a quantum system confined within the limited space of its container. According to quantum mechanics, the energy of the molecules is quantized and therefore the v_i and E_i take on discrete values, as suggested by the use of the index i. But, when the volume V occupied by the gas is large, we can approximately assume that the molecular kinetic energy $E = \frac{1}{2} m v^2$ is not quantized but varies in a *continuous* fashion. The energy distribution in the system, therefore, involves the concepts of occupation density and density of states, defined in the previous section. As can be shown [1,2], the density of states is given by the expression[4]

$$N(E) = \frac{2\pi}{h^3} (2m)^{3/2} E^{1/2} \tag{4}$$

Regarding the occupation density $n(E)$, we recall that it is defined by demanding that the product $n(E)dE$ represents the number of molecules, per unit volume, having energies between E and $E+dE$. As is found [1,2], when the gas is in statistical equilibrium,

$$n(E) = \frac{2\pi n}{(\pi kT)^{3/2}} E^{1/2} e^{-E/kT} \tag{5}$$

where n is the concentration of the molecules (number of molecules per unit volume) and T is the absolute temperature. Note that *there is no limit to the number of molecules that can occupy a given quantum state*. In other words, the molecules of the ideal gas do not obey the *Pauli exclusion principle*.

The *average* (*kinetic*) *energy* of the molecules at temperature T is given by [1,2]

$$\langle U \rangle = \frac{3}{2} kT \tag{6}$$

The constant k appearing in Eqs. (3.5) and (3.6) is called the *Boltzmann constant* and is equal to

$$k = 8.62 \times 10^{-5}\, eV/K = 1.38 \times 10^{-23}\, J/K \tag{7}$$

[4] Since the energy E is purely kinetic, we have that $E \geq 0$; thus the presence of E inside a square root is acceptable.

If N is the total number of molecules in the gas, the total energy of the system is equal to $N\langle U\rangle$. Thus, if V is the volume occupied by the gas, the *total energy per unit volume* of the system is

$$U = \frac{N}{V}\langle U\rangle = n\langle U\rangle = \frac{3}{2}\, nkT \tag{8}$$

Notice that, according to (6),

> *the absolute temperature T of an ideal gas is a measure of the average kinetic energy of the molecules in a state of statistical equilibrium.*

In particular, *the kinetic energy of the molecules vanishes at absolute zero* (T=0). As we will see in Sec. 4, an analogous statement is *not* valid for the free electrons in a metal, despite the superficial similarities of the latter system with the molecules of an ideal gas.

3. Bosons and fermions

The *Maxwell-Boltzmann theory* for an ideal gas, outlined in the previous section, is essentially a *classical* theory. Although we regarded the gas molecules as quantum particles (for example, we assumed that they occupy quantum states) the underlying analysis in essence treats the molecules as classical particles since it does not take into account one of the most important principles of quantum theory; namely, the *uncertainty principle*. [Don't be deceived by the presence of the quantum constant h in Eq. (4); the basic result (5) for the occupation density may be derived by entirely classical methods, without recourse to quantum mechanics.] Such an omission of quantum principles is not allowable in the case of electrons, given their exceedingly microscopic nature in comparison to gas molecules. The treatment of such profoundly quantum problems is the subject of quantum statistics.

In quantum statistics, *identical* particles that *interact* with one another are considered *indistinguishable*. By "identical particles" we mean particles that may replace one another without any observable effects in the macroscopic state of the system. (For example, the free electrons in a metal are identical particles since it doesn't matter *which* individual electrons occupy an energy level; it only matters *how many* electrons occupy that level.) In classical mechanics, where the notion of the trajectory of a particle is physically meaningful, it is possible to distinguish identical particles that interact by simply following the path of each particle in the course of an experiment. We say that classical particles are *distinguishable*. This is the view adopted by the Maxwell-Boltzmann theory for the molecules of ideal gases.

Things are not that simple, however, for systems of extremely microscopic particles such as, e.g., the electrons in a metal, given that the uncertainty principle does not allow a precise knowledge of the trajectories of such purely quantum particles (in quantum theory the notion of the trajectory is meaningless). Therefore, when *identical* quantum particles interact with one another, it is impossible to distinguish one from another during an experiment. We say that interacting identical particles are *indistinguishable*. (Identical particles that do *not* interact are considered distinguishable.)

Thus, quantum statistics is the enhancement of the corresponding classical theory by taking into account the implications of the uncertainty principle. According to the quantum theory, there are two kinds of fundamental particles in Nature, which follow

separate statistical laws of distribution of energy when they are grouped to form systems of identical and indistinguishable particles:

- the particles that obey the Pauli exclusion principle are called *fermions* and they follow the *Fermi-Dirac distribution law*;
- the particles that do *not* obey the Pauli exclusion principle are called *bosons* and they follow the *Bose-Einstein distribution law*

(see discussion in Appendix A). As has been observed,

> *particles having half-integer spins* (e.g., electrons) *are fermions,* while *particles with integral spins* (e.g., photons) *are bosons.*

Accordingly,

> *two or more identical fermions may <u>not</u> occupy the same quantum state in a system,* whereas *an arbitrary number of identical bosons <u>may</u> occupy the same quantum state.*

Given that even the molecules of ideal gases are quantum systems consisting of various kinds of fermions (electrons, protons, neutrons, not to mention quarks!) we may wonder whether the Maxwell-Boltzmann distribution law has any use after all. Well, what keeps the classical theory in the game is the fact that, *for systems in which the uncertainty principle can be ignored*, both Fermi-Dirac and Bose-Einstein statistics reduce to Maxwell-Boltzmann statistics. Such a semi-classical system is an ideal gas of low density (i.e., having a small concentration of molecules) at a high temperature. In this case quantum effects are not significant and the use of classical statistical methods leads to correct physical predictions.

4. Fermi-Dirac distribution law for a metal

Fermi-Dirac statistics applies to systems of identical and indistinguishable particles that obey the Pauli exclusion principle; that is, to systems of *fermions*. The free electrons in a metal are an important example of such a system. Although the energies of the electrons are quantized, we may approximately regard these energies as varying continuously within the limits of the conduction band. This approximation is valid when the volume of space within which the motion of the electrons takes place is relatively large (a similar condition is valid for the molecules of an ideal gas).

The mobile electrons in a metal are called *free* because of their ability to move in between the positive ions without being subject to forces of appreciable strength (except, of course, when the electrons accidentally collide with the ions). In general, a free particle has constant potential energy that may arbitrarily be assigned zero value. The energy E of a free electron is thus *purely kinetic,* which means that $E \geq 0$. We will therefore assume that the energy of a free electron in the metal may take on all values from 0 up to $+\infty$. (The upper limit is, of course, purely theoretical since the energy of an electron in the interior of a metal may not exceed the *work function* of that metal, equal to the minimum energy required in order that the electron may “escape” from the crystal.)

Let $N(E)$ be the density of states in the conduction band of the metal. We recall that this function is defined so that the product $N(E)dE$ is equal to the number of

quantum states (per unit volume) with energies between E and $E+dE$ (equivalently, equal to the number of states belonging to all energy levels between E and $E+dE$ in the conduction band). As can be shown [1-5] the function $N(E)$ is given by the expression

$$N(E) = \frac{4\pi}{h^3}(2m)^{3/2}E^{1/2} \equiv \gamma E^{1/2} \tag{9}$$

where m is the mass of the electron. By comparing (9) with (4) we observe that the density of states for the electrons in a metal is twice that for the molecules of an ideal gas. This is due to the two possible orientations of the electron spin, that is, the two possible values of the quantum number m_s ($=\pm ½$). This consideration does not appear in the Maxwell-Boltzmann distribution since the classical theory does not take into account purely quantum concepts such as that of the spin of a particle.

To find the distribution of energy for the free electrons in a metal we must determine the occupation density $n(E)$. As we know, this function is defined so that the product $n(E)dE$ represents the number of free electrons (per unit volume of the metal) with energies between E and $E+dE$ (equivalently, the number of electrons occupying the energy levels between E and $E+dE$ in the conduction band). Because of the Pauli exclusion principle, the number of electrons in this elementary energy interval cannot exceed the number of available quantum states in that interval:

$$n(E)\,dE \le N(E)\,dE \quad \Rightarrow \quad 0 \le \frac{n(E)}{N(E)} \le 1$$

We observe that the quotient $n(E)/N(E)$ satisfies the necessary conditions in order to represent probability (see Appendix B). We thus define the *probability function* $f(E)$:

$$f(E) = \frac{n(E)}{N(E)} \quad \Leftrightarrow \quad n(E) = f(E)\,N(E) \tag{10}$$

The function $f(E)$ represents the *fraction of states of energy E that are occupied by electrons*, or, equivalently, the *occupation probability* for any state of energy E.

The analytical expression for $f(E)$ is given by the *Fermi-Dirac distribution function*

$$\boxed{f(E) = \frac{1}{1+e^{(E-E_F)/kT}}} \tag{11}$$

where T is the absolute temperature, k is the Boltzmann constant (7), and E_F is a parameter called the *Fermi energy* (or *Fermi level*, on an energy-level diagram) for the considered metal. We note that, although the present discussion concerns free electrons in metals, the expression (11) is generally valid for *all* systems of fermions.

By combining (10), (11) and (9) we can now write an expression for the occupation density $n(E)$, which quantity determines the distribution of energy for the free electrons in the metal:

$$n(E) = f(E)N(E) = \frac{\gamma E^{1/2}}{1+e^{(E-E_F)/kT}} \tag{12}$$

The physical significance of the Fermi energy E_F can be deduced from (11) after making the following mathematical observations:

- For $T \to 0$, $\lim_{T \to 0^+} \left[e^{(E-E_F)/kT} \right] = \begin{cases} \infty, & E > E_F \\ 0, & E < E_F \end{cases}$
- For $T > 0$, $e^{(E-E_F)/kT} = 1$ when $E = E_F$

Therefore,

$$\text{for } T = 0 \quad \Rightarrow \quad f(E) = \begin{cases} 0, & E > E_F \\ 1, & E < E_F \end{cases} \tag{13}$$

while

$$\text{for } T > 0 \quad \Rightarrow \quad f(E_F) = \frac{1}{2} \tag{14}$$

These are physically interpreted as follows:

1. For T=0, *all* states with energies $E<E_F$ (i.e., all states up to the Fermi level) are *occupied* by electrons, while *all* states with $E>E_F$ are *empty*.
2. For T >0, *half* the states with energy $E=E_F$ are occupied. That is, the occupation probability of any state on the Fermi level is equal to 50%.

We notice that the function $f(E)$ is non-continuous for $E=E_F$ when T=0. Hence, the occupation probability on the Fermi level is indeterminate for T=0. Figure 2 shows the graph of $f(E)$ for T= 0 and T > 0. A diagram of this form applies, in general, to *any* system of fermions (not just to free electrons in metals).

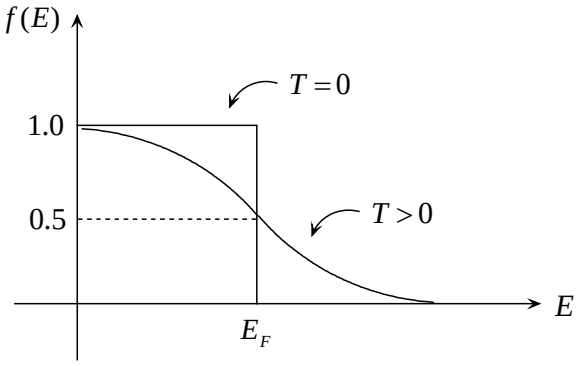

Fig. 2

As we saw, the Fermi energy E_F places an upper limit to the energies of the free electrons in a metal at T=0. Since the energy of a free electron is purely kinetic, we can write:

$$E_F = (E_{kinetic})_{max} \quad \text{for} \quad T=0 \tag{15}$$

That is,

> *the Fermi energy of a metal represents the maximum kinetic energy of the free electrons at absolute zero* ($T = 0$) .

Therefore, at T=0, all quantum states in the conduction band ranging from the lowest energy level E=0 up to the Fermi level E=E_F are occupied by the free electrons, while all states above E_F are empty. The diagram in Fig. 3 shows the conduction band of the metal for T=0.

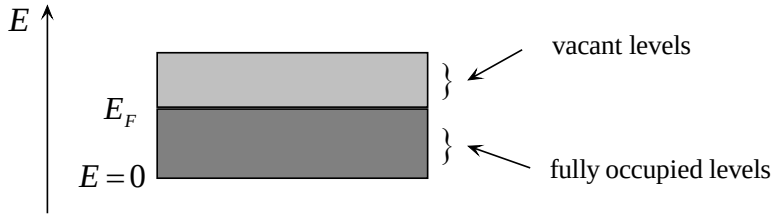

Fig. 3

We notice a fundamental difference of the Fermi-Dirac theory for electrons from the classical theory for ideal gases. According to the latter theory, all gas molecules must have zero (kinetic) energy at absolute zero. On the other hand, at T=0 the free electrons in a metal have (kinetic) energies ranging from zero up to the Fermi energy. This occurs because the electrons, being fermions, obey the Pauli exclusion principle which does not allow all of them to occupy the lowest energy level E=0, given that this level does not possess a sufficient number of quantum states to accommodate all electrons. At temperatures T >0, however, by receiving thermal energy, some free electrons acquire (kinetic) energies greater than E_F. These electrons then occupy energy levels above the Fermi level within the conduction band. As we saw, on the Fermi level itself *half* the available quantum states are occupied for T >0.

We now describe a method for determining the Fermi energy E_F of the system of mobile electrons in a metal. Let n be the *electronic density* of the metal (number of free electrons per unit volume) and let $n(E)$ be the occupation density of the Fermi-Dirac distribution. These two quantities are related by Eq. (3):

$$n = \int_{E_1}^{E_2} n(E)\,dE = \int_0^{\infty} n(E)\,dE \tag{16}$$

where here we have put E_1=0 and E_2=+∞ (a purely theoretical limit, of course!). Using the expression (12) for $n(E)$, we have:

$$n = \int_0^{\infty} \frac{\gamma E^{1/2}}{1+e^{(E-E_F)/kT}}\,dE \tag{17}$$

If we could compute the integral in (17) analytically, the only thing to do would be to solve the result for E_F and thus express the Fermi energy as a function of n and T. Since, however, handling the above integral is not an easy task, we will restrict ourselves to something much easier; namely, we will evaluate E_F for the special case where T=0. From (10), (9) and (13) we have that, at this temperature,

$$n(E) = f(E)N(E) = \begin{cases} 0\,, & E > E_F \\ \\ \gamma E^{1/2}\,, & 0 \le E < E_F \end{cases} \tag{18}$$

Substituting (18) into (16), we find

$$n = \int_0^{E_F} n(E)\,dE + \int_{E_F}^{\infty} n(E)\,dE = \int_0^{E_F} \gamma E^{1/2}\,dE + 0 \quad \Rightarrow$$

$$n = \frac{2}{3}\gamma E_F^{\,3/2} \tag{19}$$

so that

$$\boxed{E_F = \left(\frac{3n}{2\gamma}\right)^{2/3}} \tag{20}$$

We observe that the Fermi energy of the metal at T=0 depends only on the concentration n of free electrons and is independent of the dimensions of the crystal (i.e., of the total number of ions). As can be proven (see [4], Sec. 9-3) the value of E_F that we have found does not change much at higher temperatures. Thus, although derived for T=0, relation (20) will be assumed valid *for all T*. Typical values of E_F for metals range from about $3\,eV$ to $12\,eV$.

5. Fermi-Dirac distribution for an intrinsic semiconductor

In an *intrinsic* semiconductor [3-5] (i.e., one without impurities) the electronic system of interest consists of the valence electrons of the atoms; specifically, the electrons that participate in covalent bonds as well as those that are free. In terms of energy, the aforementioned two groups of electrons belong to the *valence band* and the *conduction band*, respectively. The distribution of electrons within the allowable energy states is determined by the occupation density $n(E)$, which is related to the density of states $N(E)$ and the probability function $f(E)$ by

$$n(E) = f(E)\,N(E) \tag{21}$$

As we know, the product $n(E)dE$ represents the number of electrons, per unit volume of the material, with energies between E and $E+dE$.

The form of the function $N(E)$, analogous to the expression (9) for metals, depends on the energy region within which this function is defined [3-5] (see Fig. 4):

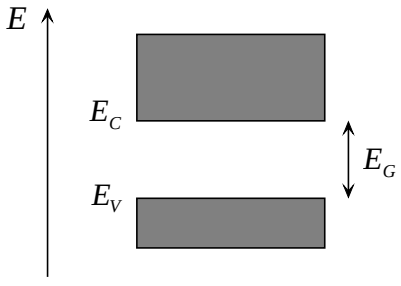

Fig. 4

a. In the *conduction band*,

$$N(E) = \gamma\,(E - E_C)^{1/2}\ , \qquad E \geq E_C \tag{22}$$

b. In the *valence band,*

$$N(E) = \gamma (E_V - E)^{1/2} \ , \qquad E \le E_V \tag{23}$$

c. In the *forbidden band* of a pure semiconductor there are no allowable quantum states; therefore,

$$N(E) = 0 \ , \qquad E_V < E < E_C \tag{24}$$

The probability function for the electrons is given by the Fermi-Dirac formula:

$$f(E) = \frac{1}{1 + e^{(E - E_F)/kT}} \tag{25}$$

where E admits values in the above-mentioned three energy regions. We would now like to find the probability function $f_p(E)$ for the *holes* in the valence band of a semiconductor. We think as follows: A quantum state at an energy level E in the valence band is either occupied by an electron or "occupied" by a hole. If $f(E)$ and $f_p(E)$ are the corresponding occupation probabilities, then

$$f(E) + f_p(E) = 1 \quad \Leftrightarrow \quad f_p(E) = 1 - f(E) \tag{26}$$

Substituting the expression (25) for $f(E)$, we find that

$$f_p(E) = \frac{e^{(E - E_F)/kT}}{1 + e^{(E - E_F)/kT}} \tag{27}$$

Physically, the function $f_p(E)$ represents the fraction of states of energy E that are *not* occupied by electrons, or, equivalently, the probability of non-occupation of a state of energy E.

6. Fermi energy in semiconductors

The Fermi energy of an *intrinsic* semiconductor [3-5] is given by Eq. (28), where the energy levels E_V and E_C were introduced in the previous section (see Fig. 5):

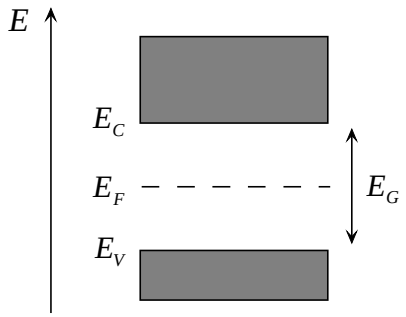

Fig. 5

$$E_F = \frac{E_V + E_C}{2} \tag{28}$$

We write:

$$E_F = \frac{E_V + (E_V + E_G)}{2} = E_V + \frac{E_G}{2} \tag{29}$$

This means that

> *the Fermi level of an intrinsic semiconductor is located at the center of the forbidden band.*

Furthermore, E_F is *independent of temperature*, as well as independent of the dimensions of the crystal (that is, of the number of atoms in the lattice).

How should we interpret the presence of E_F inside the forbidden band of a pure semiconductor? Must we conclude that there *is*, after all, some allowable energy level in an energy region that we normally consider inaccessible to the electrons? No! Generally speaking, the Fermi energy E_F is only a parameter of the Fermi-Dirac distribution law and *does not necessarily represent an allowable energy level for the electrons*. That is, the Fermi level may or may not contain allowable quantum states. In metals, E_F is an allowable energy level since it is located inside the conduction band. This is not the case for intrinsic semiconductors, where the Fermi level is located inside the forbidden band.

We note that the presence of the Fermi level E_F inside the forbidden band is consistent with the general physical interpretation of the Fermi energy given in Sec. 4. Let us explain why:

(*a*) For $T>0$ we know that $f(E_F)=1/2$. That is, half the states of the Fermi level are occupied by electrons. In our case, however, the level E_F is located inside the forbidden band; hence it may not possess allowable quantum states. Thus on the Fermi level we have the following situation:

$$\tfrac{1}{2} \times 0 \text{ states} \;\Rightarrow\; 0 \text{ electrons}$$

which is reasonable, given that no energy level inside the forbidden band of an *intrinsic* semiconductor may contain electrons.

(*b*) For $T=0$, all *allowable* energy levels below E_F are completely filled while all *allowable* levels above E_F are empty. But, allowable levels immediately below and above E_F exist in the valence and the conduction band, respectively. Hence, all levels in the valence band are fully occupied by the atomic valence electrons, while no energy level within the conduction band contains electrons. Physically this means that, for $T=0$, all covalent bonds are intact and there are no free electrons in the crystal.

The fact that the level E_F is at the center of the forbidden band reflects a symmetry between electrons and holes in an intrinsic semiconductor, their concentrations n and p being equal to each other and equal to the intrinsic concentration n_i:

$$n = p = n_i \qquad \text{(pure semiconductor)} \tag{30}$$

In a sense, the Fermi level "keeps equal distances" from the energy bands occupied by free electrons and holes, the two charge carriers being equally important in an intrinsic semiconductor.

The Fermi level of a pure semiconductor will be affected if we dope the crystal with impurities. The doping will spoil the electron-hole balance expressed by Eq. (30). In an *n*-type semiconductor the majority carriers are the electrons in the conduction band, while in a *p*-type semiconductor the majority carriers are the holes in the valence band. The Fermi level will then shift *toward the band occupied by the majority carriers* in each case. Thus, in an *n*-type semiconductor the Fermi level moves closer to the conduction band, while in a *p*-type semiconductor it moves closer to the valence band, as shown in Fig. 6.

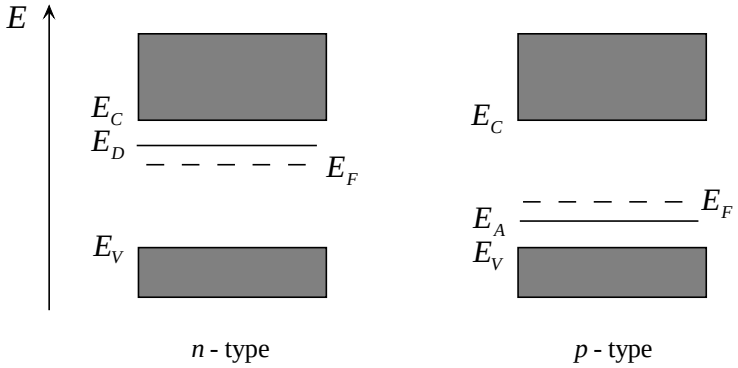

Fig. 6

In contrast to an intrinsic semiconductor, where E_F is independent of temperature (the Fermi level always lies at the center of the forbidden band), in doped semiconductors E_F changes with temperature. Specifically, as T increases, E_F moves *toward the center of the forbidden band*. This happens because, by the increase of temperature more and more covalent bonds are "broken" in the crystal, which results in an increase of concentration of intrinsic carriers (both electrons and holes) relative to the carriers contributed by the impurity atoms. The concentrations of electrons and holes thus progressively become equal and the semiconductor tends to return to its intrinsic state, with a simultaneous shift of the Fermi level toward the middle of the energy gap. Conversely, as $T\to 0$, the Fermi level E_F passes *above* the donor level E_D for *n*-type doping, or *below* the acceptor level E_A for *p*-type doping.

The value of E_F also depends on the concentration of impurity atoms. Adding more donor (acceptor) atoms in an *n*-type (*p*-type) semiconductor results in a further shift of the Fermi level toward the conduction (valence) band. In cases of extremely high doping, i.e., for $N_D > 10^{19}$ donor atoms $/cm^3$ or $N_A > 10^{19}$ acceptor atoms $/cm^3$, the Fermi level may even move into the conduction band or the valence band, respectively!

Appendix A. Symmetry properties of boson and fermion wavefunctions

As mentioned in Sec. 3, in quantum statistics particles are regarded as *identical* and *indistinguishable*. Regarding the latter property, we note that in quantum mechanics the finite size and the spreading of wave packets that describe individual particles make it impossible to distinguish between identical particles that interact with each other to an appreciable extent, in which case their wave packets overlap significantly. Hence, interacting identical particles are treated as indistinguishable.

Consider a pair of identical particles – call them 1 and 2 – and denote by $\Psi(1,2)$ the wavefunction of this system, where the notation indicates that particle 1 occupies the 'first' quantum state while particle 2 occupies the 'second' state, each state being characterized by a certain set of quantum numbers. Now, a physical quantity such as the probability density is determined by the real combination $\Psi^*\Psi=|\Psi|^2$. Since the particles are identical, the *physical* state of the system should not be altered by interchanging the particles. Thus we require that $|\Psi(1,2)|^2 = |\Psi(2,1)|^2 \Rightarrow \Psi(1,2)=\lambda\Psi(2,1)$, where λ is (generally) complex and $|\lambda|=1$.

Since λ is a property of the wavefunction, independent of the identity of the particles occupying the given states, we could equally well write $\Psi(2,1)=\lambda\Psi(1,2)$. Therefore,

$$\Psi(1,2) = \lambda\Psi(2,1) = \lambda[\lambda\Psi(1,2)] = \lambda^2\Psi(1,2) \Rightarrow \lambda^2 = 1$$

so that $\lambda=\pm1$. This means that $\Psi(1,2)=\pm\Psi(2,1)$. Two situations are possible regarding the symmetry property of the wavefunction and the corresponding identity of the quantum particles:

$\Psi(1,2) = \Psi(2,1) \Leftrightarrow$ even (*symmetric*) wavefunction $\Leftrightarrow$ *bosons* ;

$\Psi(1,2) = -\Psi(2,1) \Leftrightarrow$ odd (*antisymmetric*) wavefunction $\Leftrightarrow$ *fermions*

Consider now two identical particles 1 and 2, as well as two one-particle states ψ_a and ψ_b available for these particles. If the particles are distinguishable then it makes sense to tell with certainty which particle is in state ψ_a and which is in state ψ_b. According to quantum mechanics the wave function of the system will be either

$$\Psi(1,2) = \psi_a(1)\,\psi_b(2)$$

or

$$\Psi(1,2) = \psi_a(2)\,\psi_b(1)$$

(provided that the particles do not interact too strongly). But, if the particles are *indistinguishable*, we cannot say which particle is in which state. The wavefunction must therefore reflect this indeterminacy by allowing for all possible situations. For *identical bosons* the (normalized) wavefunction of the system is the *symmetric* combination

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right] = \Psi(2,1) \qquad \text{(A.1)}$$

while for *identical fermions* the wavefunction of the system is the *antisymmetric* combination

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right] = -\Psi(2,1) \qquad \text{(A.2)}$$

Note in particular that *two identical fermions cannot occupy the same quantum state*, which statement constitutes the *Pauli exclusion principle*. Indeed, if $\psi_a=\psi_b$ then the fermion wavefunction (A.2) is equal to $\Psi(1,2)=0 \Rightarrow |\Psi(1,2)|^2=0$, so that the probability density for this configuration is zero. The boson wavefunction (A.1), on the other hand, is nonzero for $\psi_a=\psi_b$, which reflects the fact that an *arbitrary* number of identical bosons can occupy the same quantum state [2].

Appendix B. Occupation probability for fermion states

In Sec. 4 we took it for granted that the fraction of states of energy E that are occupied by electrons equals the occupation probability for any state of energy E. We will now verify the validity of this statement.

As can be shown (see, e.g., [1]) the number of ways of distributing n identical fermions in N quantum states of energy E ($n \le N$) is given by

$$Q_{n,N} = \frac{N!}{n!\,(N-n)!}$$

(Remember that, by the Pauli exclusion principle, *each state can accommodate at most one particle*.) In particular, for n=1 and n=N we have $Q_{1,N}$ =N and $Q_{N,N}$ =1, respectively.

We want to find the probability that any given state of energy E will be occupied by a particle. The number of distributions in which this state is occupied equals the number of ways the remaining (n–1) particles can be distributed in the remaining (N–1) states. That is,

$$Q_{n-1,N-1} = \frac{(N-1)!}{(n-1)!\,[(N-1)-(n-1)]!} = \frac{(N-1)!}{(n-1)!\,(N-n)!}$$

The occupation probability P for that state is then

$$P = \frac{\text{number of distributions with given state occupied}}{\text{total number of distributions}} \quad \Rightarrow$$

$$P = \frac{Q_{n-1,N-1}}{Q_{n,N}} = \frac{n!}{(n-1)!}\,\frac{(N-1)!}{N!} = n\cdot\frac{1}{N} \quad \Rightarrow$$

$$P = \frac{n}{N} = \text{fraction of states of energy } E \text{ that are occupied}$$

In particular, for n=1 and n=N we have P=1/N and P=1, respectively.

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Galilean invariance of the work-energy theorem

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Abstract: The Galilean invariance of the work-energy theorem of Newtonian Mechanics is explicitly demonstrated.

Definition: A physical statement of Newtonian Mechanics is said to be *Galilean invariant* if it is valid with respect to all *inertial observers* (cf. Sec. 3.1 of [1]). If this statement is expressible by means of a mathematical equation, this equation must assume the *same form* in all *inertial frames of reference*.

Consider any two inertial observers O and O' with corresponding coordinate systems (or systems of axes) (x, y, z) and (x', y', z'). Let $\vec{V}$ be the velocity of O' relative to O. Clearly, this velocity is constant in time.

Consider also a particle of mass m, moving with velocity $\vec{v}$ and acceleration $\vec{a}$ with respect to O, and with velocity $\vec{v}'$ and acceleration $\vec{a}'$ with respect to O'. As shown in Sec. 2.8 of [1],

$$\begin{aligned} \vec{v}' &= \vec{v} - \vec{V} \\ \vec{a}' &= \vec{a} \end{aligned} \tag{1}$$

By Newton's 2nd law, the total force on m according to O and O' is

$$\vec{F} = d\vec{p}/dt = m\vec{a} \quad \text{and} \quad \vec{F}' = d\vec{p}'/dt = m\vec{a}',$$

respectively, where $\vec{p} = m\vec{v}$ and $\vec{p}' = m\vec{v}'$. In view of (1), then,

$$\vec{F} = \vec{F}' \tag{2}$$

Assume now that the particle m is inside a force field $\vec{F}(\vec{r})$ and moves from point A to point B along some curve in space. The inertial observers O and O' will generally perceive *different* trajectories of m from A to B. Both observers, however, define force according to Newton's 2nd law. Given that the work-energy theorem is a direct consequence of that law (see Sec. 4.3 of [1]), this theorem must be valid for both observers. That is, $W=\Delta E_k$ and, independently, $W'=\Delta E_k'$, where W is the work done on m by the field along the path AB, while $\Delta E_k = E_{k,B} - E_{k,A}$ is the change in the particle's kinetic energy along that path.

Let us now verify explicitly that, *if* $W=\Delta E_k$ for observer O, *then* $W'=\Delta E_k'$ for any other inertial observer O'.

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At time t the particle m passes through the trajectory point with position vector $\vec{r}(t)$ relative to observer O, or $\vec{r}'(t)$ relative to observer O'. By (2), both observers record the same force on m at this instant, i.e.,

$$\vec{F}'\left(\vec{r}'(t)\right) = \vec{F}\left(\vec{r}(t)\right) \quad \text{or simply} \quad \vec{F}'(t) = \vec{F}(t) \tag{3}$$

(Careful: a prime does *not* denote a derivative with respect to t!) Now, let W and W' be the works done on m from A to B according to O and O', respectively. We have:

$$W = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = \int_A^B \vec{F}\left(\vec{r}(t)\right) \cdot \frac{d\vec{r}}{dt} dt = \int_A^B \vec{F}(t) \cdot \vec{v}(t)\, dt$$

and, similarly,

$$W' = \int_A^B \vec{F}'(\vec{r}') \cdot d\vec{r}' = \int_A^B \vec{F}'(t) \cdot \vec{v}'(t)\, dt\,.$$

Taking (1) and (3) into account, we have:

$$W' = \int_A^B \vec{F}(t) \cdot \vec{v}(t)\, dt - \int_A^B \vec{F}(t) \cdot \vec{V}\, dt = W - \vec{V} \cdot \int_A^B \vec{F}(t)\, dt\,.$$

By using Newton's 2nd law, we have:

$$W' = W - m\vec{V} \cdot \int_A^B \frac{d\vec{v}}{dt} dt = W - m\vec{V} \cdot \int_A^B d\vec{v} \;\Rightarrow$$

$$W' = W - m\vec{V} \cdot \left(\vec{v}_B - \vec{v}_A\right) \tag{4}$$

On the other hand, the change in kinetic energy from A to B is, according to O,

$$\Delta E_k = \frac{1}{2} m {v_B}^2 - \frac{1}{2} m {v_A}^2$$

while according to O' and in view of (1),

$$\Delta E_k{}' = \frac{1}{2} m \left(v_B{}'\right)^2 - \frac{1}{2} m \left(v_A{}'\right)^2 \equiv \frac{1}{2} m \left(|\,\vec{v}_B{}'\,|^2 - |\,\vec{v}_A{}'\,|^2\right) = \frac{1}{2} m \left(|\,\vec{v}_B - \vec{V}\,|^2 - |\,\vec{v}_A - \vec{V}\,|^2\right).$$

By using the identity

$$|\,\vec{v} - \vec{V}\,|^2 = (\vec{v} - \vec{V}) \cdot (\vec{v} - \vec{V}) = v^2 + V^2 - 2\vec{v} \cdot \vec{V}$$

at A and B, we find:

$$\Delta E_k{}' = \frac{1}{2} m\left({v_B}^2 - {v_A}^2 - 2\vec{v}_B \cdot \vec{V} + 2\vec{v}_A \cdot \vec{V} \right) \Rightarrow$$

$$\Delta E_k{}' = \Delta E_k - m\vec{V} \cdot \left(\vec{v}_B - \vec{v}_A \right) \tag{5}$$

Subtracting (5) from (4), we have: $W' - \Delta E_k{}' = W - \Delta E_k$. So, *if* $W - \Delta E_k = 0 \Leftrightarrow W = \Delta E_k$ (i.e., if the work-energy theorem is valid in the *O*-frame) then $W' = \Delta E_k{}'$ (the theorem is valid in the *O*′-frame also). In other words, the work-energy theorem is Galilean invariant.

Exercise: Demonstrate in a similar way the Galilean invariance of the angular momentum – torque relation

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{T}$$

where $\vec{L} = m\vec{r} \times \vec{v}$ is the angular momentum of the particle *m* relative to *O*, and where $\vec{F}$ is the total force on *m* (see Sec. 3.7 of [1]).

[*Hint:* Assume that $\vec{r}' = \vec{r} - \vec{V}t$ (this means that the origins *O* and *O*′ of the two inertial frames coincide at t=0; as before, $\vec{V}$ is the constant velocity of *O*′ relative to *O*). Evaluate $\vec{L}' = m\vec{r}' \times \vec{v}'$ and, by using Newton's 2nd law, show that

$$\frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt} - t\vec{V} \times \vec{F} \tag{6}$$

Also, show that $\vec{T}' = \vec{r}' \times \vec{F}'$ is equal to

$$\vec{T}' = \vec{T} - t\vec{V} \times \vec{F} \tag{7}$$

Finally, subtract (7) from (6).]

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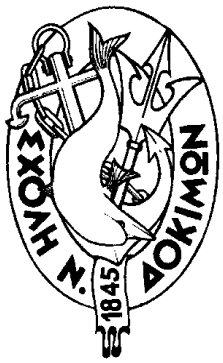
C. J. Papachristou

One-Dimensional Newtonian Systems

- **Conservative and Periodic Systems**
- **Oscillatory Systems**



Hellenic Naval Academy

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One-dimensional Newtonian systems

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The cases of conservative and oscillatory Newtonian systems in one dimension are studied. Certain unique properties of simple harmonic motion are noted.

A. One-dimensional conservative systems

1. The general solution to the problem

Consider a particle of mass m, moving along the x-axis under the action of a total force $F(x)$. The position $x(t)$ of the particle as a function of time is found by integrating the second-order differential equation (Newton's second law)

$$m\,d^2x/dt^2 = F(x) \tag{1}$$

for given initial conditions $x(t_0)=x_0$, $v(t_0)=v_0$, where $v=dx/dt$ is the velocity of the particle.

Define the auxiliary function $U(x)$ (potential energy of the particle) by

$$U(x) = -\int_0^x F(x')dx' \quad \Leftrightarrow \quad F(x) = -\,dU/dx \tag{2}$$

Then (1) is written

$$m\,d^2x/dt^2 + dU/dx = 0\,.$$

We multiply by $v=dx/dt$, which plays the role of an integrating factor:

$$(dx/dt)\,(m\,d^2x/dt^2 + dU/dx) = 0\,.$$

By noticing that

$$(dx/dt)\,(m\,d^2x/dt^2) = v\,(m\,dv/dt) = (d/dt)\,(m\,v^2/2)$$

and that $(dx/dt)\,(dU/dx) = dU/dt$, we have: $(d/dt)\,(m\,v^2/2 + U) = 0 \;\Rightarrow$

$$m\,v^2/2 + U(x) \equiv T + U = E = const. \tag{3}$$

(where T = kinetic energy) which expresses conservation of total mechanical energy.

From relation (3) we get

$$(dx/dt)^2 = (2/m)\,[E - U(x)] \;\Rightarrow\; dx/dt = \pm\,\{(2/m)\,[E-U(x)]\}^{1/2}\,.$$

Integrating this first-order differential equation and taking into account the initial condition $x=x_0$ for $t=t_0$, we have:

$$\int_{x_0}^{x} \frac{\pm\, dx}{\left\{\frac{2}{m}\left[E - U(x)\right]\right\}^{1/2}} = t - t_0 \tag{4}$$

where the plus sign is chosen for motion in the *positive* direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the *negative* direction ($v<0$, $x<x_0$).

The value of the constant E may be determined by applying the given initial conditions to (3):

$$E = m v_0^2/2 + U(x_0) \tag{5}$$

(although, as we will see, other physical considerations may also be used).

2. The case of periodic motion

Let us now assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x\neq 0$ (this arrangement is always possible because of the arbitrariness in the definition of the zero-level of the potential energy). In general, the graph of $U(x)$ need not be symmetric with respect to the axis $x=0$.

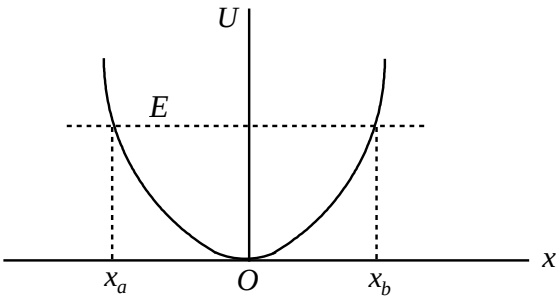

Fig. 1

Let E be the total mechanical energy of the particle. Since $E=T+U$ with $T\geq 0$, it follows that $E\geq U(x)$ for any physical motion. The motion is thus *bounded* between the points x_a and x_b of the x-axis, these points being *turning points* at which the particle stops momentarily ($E=U \Rightarrow T=0 \Rightarrow v_a=v_b=0$). The time it takes for a complete journey from x_a to x_b and back to x_a is found by using (4) with the appropriate sign for each direction of motion:

$$P = \int_{x_a}^{x_b} \frac{dx}{\{\ldots\}^{1/2}} + \int_{x_b}^{x_a} \frac{-dx}{\{\ldots\}^{1/2}} \Rightarrow$$

$$P = 2\int_{x_a}^{x_b} \frac{dx}{\left\{\frac{2}{m}\left[E - U(x)\right]\right\}^{1/2}} \tag{6}$$

Since P is fixed for given x_a and x_b , the motion is *periodic* with *period* P. Generally, the period depends on the limits of integration x_a and x_b and therefore it depends on the total energy E of the particle. An exception where P does *not* depend on E is *simple harmonic motion*, as we now show.

3. Simple harmonic motion (SHM)

In SHM the potential energy is of *parabolic* form: $U(x)=kx^2/2$, which is symmetric with respect to the axis $x=0$ (see Fig. 1). The total force is a *restoring force* given by

$$F(x) = -\,dU/dx = -kx \tag{7}$$

If frictional (damping) forces are present, the total force also contains a velocity-dependent term $-\lambda v = -\lambda dx/dt$ and the system is no longer conservative.

According to Fig. 1 the motion takes place between $x_a = -A$ and $x_b = A$, where $A \geq 0$ is the *amplitude* of oscillation. At the two extreme points the kinetic energy T vanishes momentarily and the total energy, which is equal to $E=T+U$ and which retains a fixed value during the motion, is equal to the potential energy: $E=U(\pm A)=kA^2/2$. Since E is the same at all points x, we conclude that

$$E = m v^2/2 + k x^2/2 = k A^2/2 \tag{8}$$

The period of oscillation is found by using (6):

$$P = 2\int_{-A}^{A} \left\{ \frac{2}{m}(E - kx^2/2) \right\}^{-1/2} dx\,.$$

Substituting for E from (8), we find:

$$P = \frac{2}{\omega}\int_{-A}^{A} \left(A^2 - x^2 \right)^{-1/2} dx$$

where we have set $\omega=(k/m)^{1/2}$ (angular frequency). Putting $x/A=u$ and using the integral formula

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

we finally find (see Appendix):

$$P = 2\pi/\omega = 2\pi\,(m/k)^{1/2}\,.$$

We conclude that, if the potential energy is of parabolic form: $U(x)=kx^2/2$, the period P of motion is independent of the amplitude A, thus independent of the total energy $E=kA^2/2$.

But, what if $U(x)$ is like that in Fig. 1 but *not* parabolic? For example, let U be of the form $U(x)=\lambda x^4/4$, so that $F(x) = -dU/dx = -\lambda x^3$. Since $U(x)$ is symmetric with respect to the axis $x=0$, the periodic motion will take place between the points $x_a = -A$ and $x_b = A$ and the total energy will be equal to $E=U(\pm A)=\lambda A^4/4$. The period is

$$P = 2\int_{-A}^{A} \left\{ \frac{2}{m}(E - \lambda x^4/4) \right\}^{-1/2} dx = \frac{2}{\mu}\int_{-A}^{A} \left(A^4 - x^4 \right)^{-1/2} dx = \frac{2}{\mu A}\int_{-1}^{1} \frac{du}{\sqrt{1-u^4}}$$

where we have set $u=x/A$ and $\mu=(\lambda/2m)^{1/2}$. Obviously, P depends on the amplitude A, thus on the total energy E. (A more general proof regarding non-parabolic potential energies, in general, is given in the Appendix.)

Returning to SHM, we may obtain the equation of motion $x=x(t)$ by using (4) with $U(x)=kx^2/2$ and $E=kA^2/2$. Let us assume first that the motion is in the positive direction, so that $x>x_0$. Setting $\omega=(k/m)^{1/2}$, we have:

$$\int_{x_0}^{x} \left(A^2 - x^2\right)^{-1/2} dx = \omega\,(t - t_0)\,.$$

Using the integral formula

$$\int \left(A^2 - x^2\right)^{-1/2} dx = \arcsin(x/A) + C$$

and making appropriate substitutions for constants, we find an equation of the form[1]

$$\arcsin(x/A) = \omega t+\alpha \quad \Rightarrow \quad x = A\sin(\omega t+\alpha)\,.$$

For motion in the negative direction ($x<x_0$) we choose the minus sign in (4), so that

$$\int_{x_0}^{x} \left(A^2 - x^2\right)^{-1/2} dx = -\,\omega\,(t - t_0)\,.$$

This yields a result of the form[2]

$$\arcsin(x/A) = -\,\omega t+\beta \quad \Rightarrow \quad x = -A\sin(\omega t-\beta)\,.$$

Since the constant β is arbitrary (being dependent on the arbitrary constants x_0 and t_0) we may set $-\beta \equiv \pi+\alpha$, so that $x = A\sin(\omega t+\alpha)$, as before.

Thus, the general solution for SHM is $x(t) = A\sin(\omega t+\alpha)$. Physically, A is the *amplitude* of oscillation, ω is the *angular frequency* and α is the *initial phase* (i.e., the *phase* $\omega t+\alpha$ at $t=0$).

4. Motion under a constant force of gravity

A projectile of mass m is fired straight upward at time $t_0=0$ from the point $x=0$ of the vertical x-axis, with initial velocity $v_0>0$ (we choose the positive direction of the x-axis to be upward). The constant acceleration of gravity is directed downward, so that $a=dv/dt=-g$. The total force on the particle (assuming no air resistance) and the corresponding potential energy of the particle are given by

$$F(x) = ma = -mg \quad \Leftrightarrow \quad U(x) = mgx \quad [\text{we assume that } U(0)=0]\,.$$

Relation (4) (with the plus sign for upward motion) is written

$$\int_0^x \frac{dx}{(E - mgx)^{1/2}} = (2/m)^{1/2}\,t\ .$$

[1] Explicitly: $\alpha = \arcsin(x_0/A) - \omega t_0$.
[2] Explicitly: $\beta = \arcsin(x_0/A) + \omega t_0$.

By (5) and by using the initial conditions we have that $E=mv_0^2/2+U(0)=mv_0^2/2$ (since $U=0$ for $x_0=0$). Thus, the requirement $E-mgx \geq 0$ yields $x \leq v_0^2/2g$. Physically this means that the particle will reach a maximum height $h=v_0^2/2g$ where it will stop momentarily before it starts to move downward (i.e., in the negative direction).

With this restriction on the acceptable values of x, the integration may be performed to give

$$(E - mgx)^{1/2} = E^{1/2} - (m/2)^{1/2}\, g\, t\ .$$

Squaring this, we find:

$$x = (2E/m)^{1/2}\, t - g\, t^2/2\ .$$

But, $E = m\, v_0^2/2 \ \Rightarrow (2E/m)^{1/2} = v_0$ (since $v_0 > 0$). Thus, finally,

$$x = v_0\, t - g\, t^2/2$$

which is, of course, a familiar result.

5. Phase curves of a one-dimensional conservative system

Newton's law for one-dimensional motion: $md^2x/dt^2=F(x)$, a second-order differential equation, may be rewritten as a system of first-order equations:

$$dx/dt = v\ , \qquad m\, dv/dt = F(x) \tag{9}$$

Dividing these equations in order to eliminate dt, we have:

$$m\, v\, dv = F(x)\, dx = -\, dU$$

where

$$U(x) = -\int_0^x F(x')dx' \ \Leftrightarrow \ F(x) = -\, dU/dx\ .$$

Thus, $m\, v\, dv + dU = d\,(m\, v^2/2 + U) = 0 \ \Rightarrow$

$$m\, v^2/2 + U(x) = E \equiv const. \tag{10}$$

For each value of the constant E (total energy), Eq. (10) defines a curve in the 2-dimensional *phase space* with coordinates (x, v). This curve is called a *phase curve*. The value of E is uniquely determined by the initial conditions of the system, according to (5). Since the solution of the system (9) is unique for given initial conditions, *no two phase curves may intersect* in phase space. Let us see two examples:

1. Simple harmonic motion (cf. Sec. 3)

Conservation of mechanical energy in SHM is expressed by $mv^2/2 + kx^2/2 = E \ \Rightarrow$

$$\frac{x^2}{2E/k} + \frac{v^2}{2E/m} = 1 \qquad \text{(equation of an } \textit{ellipse}\text{)}$$

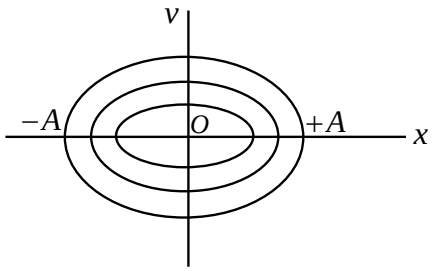

Fig. 2

Figure 2 shows a family of ellipses in phase space, corresponding to different values of E. Notice that, for $v=0 \Rightarrow x= \pm(2E/k)^{1/2} \equiv \pm A$, so that $E=kA^2/2$. Note also that the equations of motion, $\{dx/dt = v,\ dv/dt = -kx/m\}$, endow the phase curves with a sense of direction for increasing t (i.e., for $dt>0$). Indeed, the velocity v is positive (negative) for increasing (decreasing) x, while v decreases (increases) *algebraically* for positive (negative) x. This indicates that the phase curves are described *clockwise*.

2. Vertical motion under the force of gravity (cf. Sec. 4)

Conservation of mechanical energy is expressed by $mv^2/2 + mgx = E \ \Rightarrow$

$$v^2 = (2/m)(E - mgx) \quad \text{(equation of a } \textit{parabola}\text{)}$$

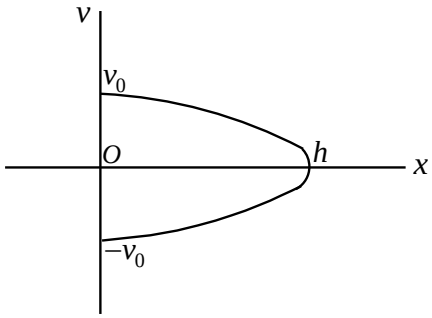

Fig. 3

Since $v^2 \geq 0$, we must have $E - mgx \geq 0 \Rightarrow x \leq E/mg$. Physically, this means that the particle will reach a maximum height $h=E/mg$ where it will stop momentarily and then its direction of motion will be reversed. On the other hand, at $x=0$ the velocity is $\pm v_0$ (see Fig. 3) where $v_0^2=2E/m \Rightarrow E=mv_0^2/2$. The maximum height is thus $h=v_0^2/2g$.

B. Oscillatory motion of (generally) non-conservative systems

1. Second-order linear differential equations

A second-order linear differential equation (DE) has the general form

$$y'' + a(x)\,y' + b(x)\,y = f(x) \tag{1}$$

where $y=y(x)$ and where $a(x)$, $b(x)$, $f(x)$ are given functions. If $f(x)\equiv 0$, the DE (1) is called *homogeneous linear*:

$$y'' + a(x)\,y' + b(x)\,y = 0 \tag{2}$$

As is easy to prove, if a function $y_1(x)$ is a solution of (2), then so is the function $y_2(x)=Cy_1(x)$ (C=const.). More generally, the following is true:

Theorem 1: If $y_1(x)$, $y_2(x)$,… are solutions of the homogeneous DE (2), then every linear combination of the form $y=C_1\,y_1(x)+C_2\,y_2(x)+\ldots$ (where C_1 , C_2 ,… are constants) also is a solution of (2).

Proof: By substituting for y on the left-hand side of (2) and by taking into account that each of the $y_1(x)$, $y_2(x)$,… satisfies this DE, we have:

$$y'' + a(x)\,y' + b(x)\,y = C_1\,(y_1'' + a\,y_1' + b\,y_1) + C_2\,(y_2'' + a\,y_2' + b\,y_2) + \ldots = 0\ .$$

Let $y_1(x)$ and $y_2(x)$ be two non-vanishing solutions of the homogeneous DE (2) [notice that the zero function $y(x)\equiv 0$ is a particular solution of (2)]. We say that the functions y_1 and y_2 are *linearly independent* if one is not a scalar multiple of the other. To put it in more formal terms, linear independence of y_1 and y_2 means that a relation of the form $C_1\,y_1(x)+C_2\,y_2(x)\equiv 0$ can only be true if $C_1=C_2=0$.

If we manage to find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous DE (2) (I can assure you that no other solution linearly independent of the former two exists!) then the *general solution* of (2) is the linear combination

$$y = C_1\,y_1(x) + C_2\,y_2(x) \tag{3}$$

where C_1 , C_2 are arbitrary constants.

Theorem 2: The general solution of the non-homogeneous DE (1) is the sum of the general solution (3) of the corresponding homogeneous equation (2) and *any particular solution* of (1).

Analytically: Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the homogeneous DE (2), and let $y_0(x)$ be any particular solution of (1). Then, the general solution of (1) is

$$y = C_1\,y_1(x) + C_2\,y_2(x) + y_0(x) \tag{4}$$

This practically means that, for any chosen y_0 , any other particular solution of (1) can be derived from (4) by properly choosing the constants C_1 and C_2 . Since (4) contains the totality of particular solutions of (1), it must be the general solution of (1).

2. Homogeneous linear equation with constant coefficients

This DE has the form

$$y'' + a\,y' + b\,y = 0 \tag{5}$$

with constant a and b. It will be assumed that a and b are real numbers.

Theorem 3: If the complex function $y=u(x)+iv(x)$ satisfies the DE (5), then the same is true for each of the real functions $y_1=u(x)$ and $y_2=v(x)$ (real and imaginary part of y, respectively).

Proof: Putting $y=u+iv$ into (5), we find:

$$(u'' + a\,u' + b\,u) + i\,(v'' + a\,v' + b\,v) = 0\ ,$$

which is true iff $u''+a\,u'+b\,u = 0$ and $v''+a\,v'+b\,v = 0$.

The standard method for solving (5) is the following: We try an exponential solution of the form $y=e^{kx}$. Then, $y'=ke^{kx}$, $y''=k^2e^{kx}$, and (5) yields (after eliminating e^{kx}):

$$k^2 + ak + b = 0 \quad (\textit{characteristic equation}) \tag{6}$$

We distinguish the following cases:

1. Eq. (6) has real and distinct roots k_1 , k_2 . Then, the functions e^{k_1x} and e^{k_2x} are linearly independent and, according to (3), the general solution of (5) is of the form

$$y = C_1\,e^{k_1x} + C_2\,e^{k_2x} \tag{7}$$

2. Eq. (6) has real and equal roots, $k_1 = k_2 \equiv k$. The general solution of (5) is, in this case (check!),

$$y = (C_1 + C_2\,x)\,e^{kx} \tag{8}$$

3. Eq. (6) has complex conjugate roots $k_1=\alpha+i\beta$, $k_2=\alpha-i\beta$ (where α, β are real). The general solution of (5) is

$$y = C_1\,e^{k_1x} + C_2\,e^{k_2x} = e^{\alpha x}\,(C_1\,e^{i\beta x} + C_2\,e^{-i\beta x})\ .$$

By Euler's formula, $e^{\pm i\beta x} = \cos\beta x \pm i \sin\beta x$. We thus have:

$$y = e^{\alpha x}\,[(C_1 + C_2)\cos\beta x + i\,(C_1 - C_2)\sin\beta x]\ .$$

Since the (generally complex) constants C_1 and C_2 are arbitrary, we may put C_1 in place of C_1+C_2 and C_2 in place of $i\,(C_1-C_2)$, so that, finally,

$$y = e^{\alpha x}\,(C_1\cos\beta x + C_2\sin\beta x) \tag{9}$$

In any case, the general solution of (5) contains two arbitrary constants C_1 and C_2 . Upon assigning specific values to C_1 and C_2 we get a *particular solution* of (5). The values of C_1 and C_2 (and thus the particular solution itself) are determined from the

general solution if we are given two *initial conditions* that the sought-for particular solution must obey. There are two kinds of initial conditions:

(*a*) We are given the values of $y(x)$ and $y'(x)$ for some value $x=x_0$ of x.

(*b*) We are given the values of $y(x)$ for $x=x_1$ and $x=x_2$.

Examples:

1. $y'' - y' - 2y = 0 \Rightarrow a=-1,\ b=-2$. The characteristic equation (6) is written: $k^2 - k - 2 = 0$, with real roots $k_1=2,\ k_2=-1$. The general solution (7) is $y = C_1 e^{2x} + C_2 e^{-x}$. Assume the initial conditions $y=2$ and $y'=-5$ when $x=0$. Then, $C_1=-1$, $C_2=3$ (show it!) and we get the *particular* solution $y = -e^{2x} + 3e^{-x}$.

2. $y'' - 6y' + 9y = 0 \Rightarrow a=-6,\ b=9$. The characteristic equation (6) is written: $k^2 - 6k + 9 = 0$, with real and equal roots $k_1=k_2=3$. The general solution (8) is $y = (C_1 + C_2 x)e^{3x}$.

3. $y'' - 4y' + 13y = 0 \Rightarrow a=-4,\ b=13$. The characteristic equation (6) is written: $k^2 - 4k + 13 = 0$, with complex conjugate roots $k_1=2+3i,\ k_2=2-3i$. The general solution (9) is (with $\alpha=2,\ \beta=3$): $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$. (Show that essentially the same result is found by making the alternative choice $\alpha=2,\ \beta=-3$.)

3. Harmonic oscillation

In a harmonic oscillation along the x-axis the total force on the oscillating body (of mass m) is $F=-kx$ ($k>0$), where x is the momentary displacement of the body from the position of equilibrium ($x=0$). By Newton's second law we have that $F=ma$, where a is the acceleration of the body: $a=d^2x/dt^2$. Therefore,

$$m\,d^2x/dt^2 = -kx$$

or, setting $k/m \equiv \omega^2$ (where we assume that $\omega>0$),

$$x'' + \omega^2 x = 0 \tag{10}$$

Eq. (10) is a homogeneous linear DE of the form (5) with x in place of y and t in place of x (notice that the first-derivative term is missing in this case). The characteristic equation (6) is written: $k^2+\omega^2=0$ (or, analytically, $k^2+0k+\omega^2=0$), with complex roots $k=\pm i\omega$ (analytically, $k_1=0+i\omega$, $k_2=0-i\omega$). The general solution of (10) is given by (9), with $\alpha=0$ and $\beta=\omega$:

$$x = C_1 \cos \omega t + C_2 \sin \omega t \tag{11}$$

where we assume that the constant coefficients C_1 and C_2 are real in order for the solution (11) to have physical meaning.

The general solution (11) can be put in different but equivalent form by setting

$$C_1 = A\sin\varphi\ ,\ \ C_2 = A\cos\varphi\ \ (A>0)\ \ \Leftrightarrow\ \ A=(C_1^{\,2}+C_2^{\,2})^{1/2}\ ,\ \ \tan\varphi = C_1/C_2\ .$$

Then,

$$x = A\sin(\omega t + \varphi) \tag{12}$$

The positive constant A is called the *amplitude* of the oscillation, while the angle φ is called the *initial phase* (the value of the *phase* $\omega t+\varphi$ at time $t=0$). The positive constant ω is the *angular frequency* of oscillation, to be called just "*frequency*" in the sequel.

Notice that, if we set $C_1=A\cos\varphi,\ C_2=-A\sin\varphi$ in (11), we will get the general solution of (10) in the form

$$x = A\cos(\omega t + \varphi) \tag{13}$$

which is equivalent to (12). Indeed, equation (13) follows directly from (12) by putting $\varphi+(\pi/2)$ in place of φ (which is arbitrary anyway) in the latter equation.

4. Damped oscillation

In a damped oscillation, in addition to the restoring force $-kx$, opposite to the displacement x from the equilibrium position, there is a frictional force $-\lambda v = -\lambda dx/dt$ ($\lambda>0$) opposite to the velocity v. The total force on the body is[3] $F = -kx-\lambda dx/dt$. By Newton's law, $F=m\,d^2x/dt^2$. Hence,

$$m\,d^2x/dt^2 = -kx - \lambda\,dx/dt\ .$$

We set

$$k/m \equiv \omega_0^{\,2}\ \ (\omega_0 = \textit{natural frequency of oscillation}\ \text{without damping}),\ \ \ \lambda/m \equiv 2\gamma,$$

so that

$$x'' + 2\gamma x' + \omega_0^{\,2} x = 0 \tag{14}$$

Eq. (14) is a homogeneous linear DE. The characteristic equation (6) is

$$k^2 + 2\gamma k + \omega_0^{\,2} = 0 \quad\Rightarrow\quad k = -\gamma \pm (\gamma^2 - \omega_0^{\,2})^{1/2}\ .$$

We distinguish the following cases:

1. *Large damping* $\Leftrightarrow\ \gamma > \omega_0$. We have two real solutions:

$$k_1 = -\gamma + (\gamma^2 - \omega_0^{\,2})^{1/2},\quad k_2 = -\gamma - (\gamma^2 - \omega_0^{\,2})^{1/2}\ .$$

The general solution of (14) is of the form (7):

$$x = C_1\,e^{k_1 t} + C_2\,e^{k_2 t} \tag{15}$$

[3] Note that a velocity-dependent force is *not* conservative. Thus, conservation of energy methods do not apply in this case.

Let us assume that $C_1>0$ and $C_2>0$. Given that $k_1<0$ and $k_2<0$ (why?) we see that $x>0$ at all times t και, moreover, $x\to 0$ as $t\to\infty$. That is, as the time t increases, the moving object approaches the equilibrium position $x=0$ without ever crossing it. The motion is therefore *non-oscillatory*.

2. *Critical damping* $\Leftrightarrow$ $\gamma = \omega_0$. Then, $k_1 = k_2 = -\gamma$, and the general solution of (14) is of the form (8):

$$x = (C_1 + C_2\, t)\, e^{kt} = (C_1 + C_2\, t)\, e^{-\gamma t} \tag{16}$$

If we assume that $C_1>0$ and $C_2>0$, we see again that $x>0$ at all t and that $x\to 0$ as $t\to\infty$. (For the term $t\, e^{-\gamma t} = t / e^{\gamma t}$ we may use L'Hospital's rule for the indeterminate form ∞/∞; show this!) Thus, there is no oscillation in this case either.

3. *Small damping* $\Leftrightarrow$ $\gamma < \omega_0$. We have two complex conjugate solutions:

$$k = -\gamma \pm i\,\omega_1 \quad \text{where} \quad \omega_1 = (\omega_0{}^2 - \gamma^2)^{1/2}.$$

The general solution will be of the form (9), with $\alpha = -\gamma$ and $\beta = \omega_1$:

$$x = e^{-\gamma t}(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t),$$

or, by setting $C_1 = A\sin\varphi$, $C_2 = A\cos\varphi$ ($A>0$),

$$x = A\, e^{-\gamma t} \sin(\omega_1 t + \varphi) \tag{17}$$

We notice that the amplitude $Ae^{-\gamma t}$ decreases exponentially with time (Fig. 1). Thus, strictly speaking, damped oscillatory motion is *not* periodic.

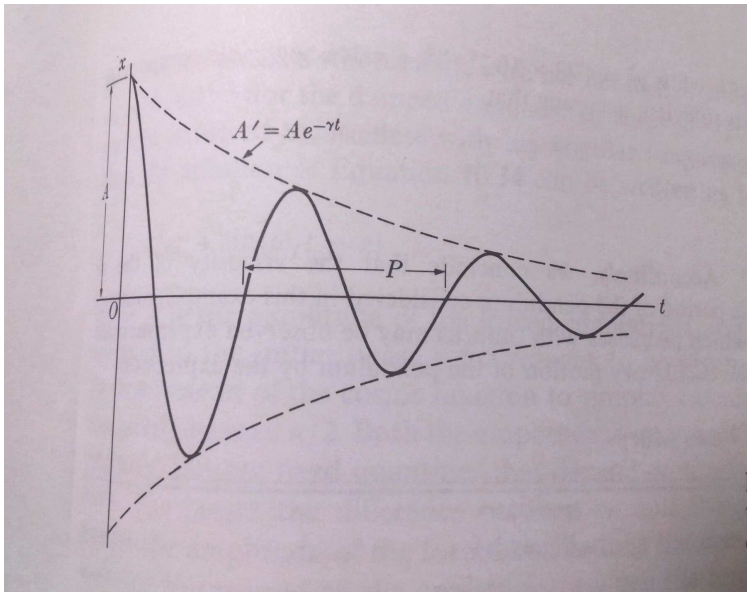

Fig. 1

5. Forced oscillation

In a forced oscillation, in addition to the restoring force $-kx$ and the frictional force $-\lambda v = -\lambda dx/dt$ the body is subject to an external force of the form

$$F(t) = F_0 \sin \omega_f t \quad (F_0 > 0) .$$

The total force on the body is $F = -kx - \lambda dx/dt + F_0 \sin \omega_f t$. By Newton's law we have that

$$m\, d^2x/dt^2 = -kx - \lambda\, dx/dt + F_0 \sin \omega_f t .$$

We set

$$k/m \equiv \omega_0^2 \ (\omega_0 = \textit{natural frequency}), \quad \lambda/m \equiv 2\gamma, \quad F_0/m \equiv f_0 ,$$

so that

$$x'' + 2\gamma x' + \omega_0^2 x = f_0 \sin \omega_f t \tag{18}$$

Eq. (18) is a non-homogeneous linear DE. According to Theorem 2 of Sec. 1, its general solution is the sum of the general solution of the corresponding homogeneous equation,

$$x'' + 2\gamma x' + \omega_0^2 x = 0 ,$$

and *any particular solution* of (18). For small damping ($\gamma < \omega_0$) the general solution of the homogeneous equation is given by (17):

$$x = A_1 e^{-\gamma t} \sin (\omega_1 t + \varphi_1) \quad \text{where} \quad \omega_1 = (\omega_0^2 - \gamma^2)^{1/2} .$$

As can be verified, a particular solution of (18) is the following:

$$x = A \sin (\omega_f t + \varphi) \tag{19}$$

where

$$A = \frac{f_0}{\left[\left(\omega_f^2 - \omega_0^2\right)^2 + 4\gamma^2 \omega_f^2\right]^{1/2}} \quad \text{and} \quad \tan \varphi = \frac{2\gamma \omega_f}{\omega_f^2 - \omega_0^2} \tag{20}$$

The general solution of (18) is, therefore,

$$x = A_1 e^{-\gamma t} \sin (\omega_1 t + \varphi_1) + A \sin (\omega_f t + \varphi) \tag{21}$$

with *arbitrary* A_1 , φ_1 . The first term on the right in (21) decreases exponentially with time and dies out quickly. In a steady-state situation, therefore, what remains is the particular solution (19):

$$x = A \sin (\omega_f t + \varphi) .$$

The amplitude A of oscillation is a function of the applied frequency ω_f, according to (20). This amplitude attains a maximum value when the denominator in the first relation (20) becomes minimum. This occurs when

$$\omega_f = (\omega_0^2 - 2\gamma^2)^{1/2} \equiv \omega_A \tag{22}$$

Proof: We set $\omega_f \equiv \omega$, for simplicity, and we consider the function

$$\Psi(\omega) = (\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2 ,$$

so that $A = f_0 / [\Psi(\omega)]^{1/2}$. We can show that

$$\Psi'(\omega) = 0 \text{ for } \omega = (\omega_0^2 - 2\gamma^2)^{1/2} = \omega_A \text{ and } \Psi''(\omega_A) = 8\omega_A^2 > 0 .$$

Thus, for small damping ($2\gamma^2 < \omega_0^2$) the function $\Psi(\omega)$ is *minimum*, hence the amplitude A is *maximum*, when $\omega_f = \omega_A$. This situation is called *amplitude resonance*.

In Fig. 2 it is assumed that $\lambda_1 < \lambda_2 \Leftrightarrow \gamma_1 < \gamma_2$. This means that, in accordance with (22), $\omega_{A,1} > \omega_{A,2}$. In the case of no damping ($\lambda = 0 \Leftrightarrow \gamma = 0$) Eq. (22) yields $\omega_A = \omega_0$. In other words, in an *undamped* forced oscillation the amplitude becomes maximum (in fact, infinite) when the applied frequency ω_f is equal to the natural frequency ω_0 of oscillation.

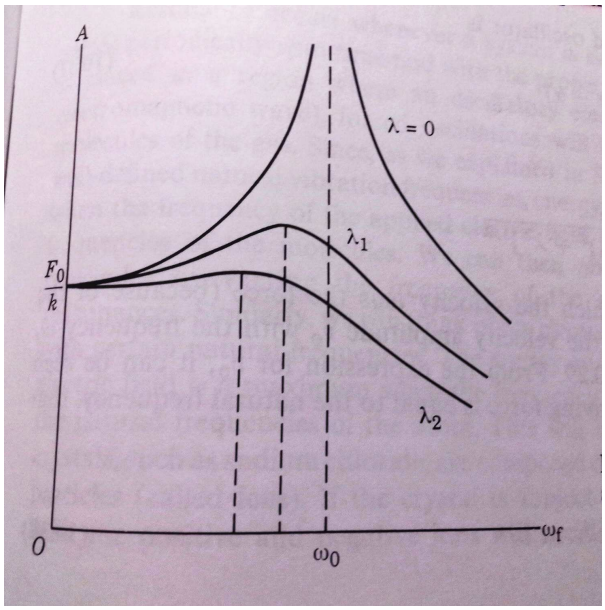

Fig. 2

By differentiating (19) we find the velocity of the oscillating body:

$$v = dx/dt = \omega_f A \cos(\omega_f t + \varphi) \equiv v_0 \cos(\omega_f t + \varphi)$$

where, by (20),

$$v_0 = \omega_f A = \frac{f_0}{\left[\left(1-\frac{\omega_0^{\,2}}{\omega_f^{\,2}}\right)^2 + 4\gamma^2\right]^{1/2}} \ .$$

The velocity amplitude v_0 becomes maximum when the denominator on the right is minimum, which occurs for $\omega_f = \omega_0$ (Fig. 3). The kinetic energy $mv_0^{\,2}/2$ then reaches its maximum value and there is *energy resonance*.

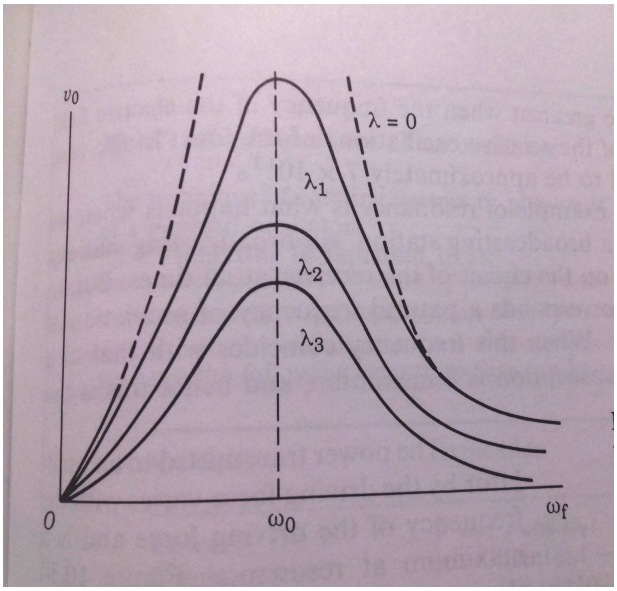

Fig. 3

Note that, in contrast to amplitude resonance, the frequency ω_f for energy resonance is independent of the damping factor λ and is always equal to the *natural frequency* ω_0 of the oscillator. At this frequency the work supplied by the external force $F(t)$ to the oscillator per unit time is maximum. That is, the oscillator absorbs the largest possible power from the external agent that exerts the force F.

Notice also that, in the case of zero damping ($\lambda=0 \Leftrightarrow \gamma=0$) the velocity amplitude v_0 becomes *infinite* at energy resonance, i.e., for $\omega_f = \omega_0$. This rather unphysical situation is, of course, purely theoretical since a mechanical motion with no friction whatsoever is practically impossible!

Appendix: Amplitude dependence of period

As we have shown, the general solution to the one-dimensional conservative Newtonian problem is

$$\int_{x_0}^{x} \frac{\pm\, dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = t - t_0 \tag{1}$$

where the plus sign is chosen for motion in the positive direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the negative direction ($v<0$, $x<x_0$).

Let us assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x\neq 0$. The graph of $U(x)$ is assumed to be symmetric with respect to the axis $x=0$, which means that $U(x)$ is an *even* function: $U(-x)=U(x)$.

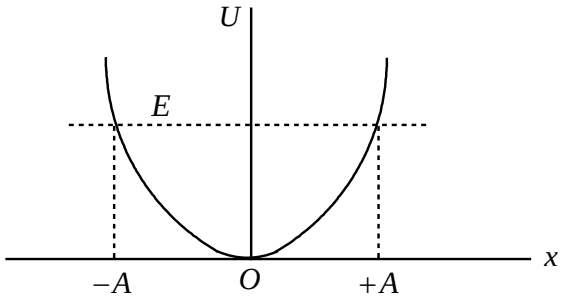

Fig. 1

If E is the total mechanical energy of the particle, then, according to Fig. 1, the motion is bounded between the points $-A$ and $+A$ of the x-axis, which are turning points at which the particle stops momentarily. Since E is constant, its value at all points equals its value at the turning points; i.e.,

$$E = U(\pm A) \tag{2}$$

The time it takes for a complete journey from $-A$ to $+A$ and back to $-A$ is found by using (1) with the appropriate sign for each direction of motion:

$$P = \int_{-A}^{A} \frac{dx}{\{\ldots\}^{1/2}} + \int_{A}^{-A} \frac{-dx}{\{\ldots\}^{1/2}} \;\Rightarrow$$

$$P = 2\int_{-A}^{A} \frac{dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = \left(2m\right)^{1/2} \int_{-A}^{A} \left[E-U(x)\right]^{-1/2} dx \tag{3}$$

Since P is fixed for a given A, the motion is periodic about the point $x=0$, with amplitude equal to A and with period P. It follows from (2) and (3) that the period P depends on A and thus on the total energy E of the particle. We will now show that an exception where P does *not* depend on A (thus on E also) is simple harmonic motion.

Since $U(x)$ is an even function with $U(0)=0$, it can be expanded into a Maclaurin series of the form

$$U(x)=\sum_{l=1}^{\infty} a_l\, x^{2l} \tag{4}$$

where the coefficients a_l are not necessarily all different from zero. From (2) we have

$$E=U(\pm A)=\sum_{l=1}^{\infty} a_l\, A^{2l}$$

so that

$$E-U(x)=\sum_{l=1}^{\infty} a_l\left(A^{2l}-x^{2l}\right).$$

Equation (3) then yields

$$P=\left(2m\right)^{1/2}\int_{-A}^{A}\left[\sum_{l=1}^{\infty} a_l\left(A^{2l}-x^{2l}\right)\right]^{-1/2} dx\,.$$

By setting $x/A=u \;\Leftrightarrow\; x=Au$, we get:

$$P=\left(2m\right)^{1/2} A\int_{-1}^{1}\left[\sum_{l=1}^{\infty} a_l A^{2l}\left(1-u^{2l}\right)\right]^{-1/2} du \tag{5}$$

It is obvious that, in general, P depends on A. The only exception where P is *not* dependent on A is the case where the following condition is satisfied: $a_l=0$ for $l\neq 1$. That is, the only nonvanishing coefficient a_l in the series (4) is a_1. By setting $a_1=k/2$ the potential energy (4) reduces to $U(x)=kx^2/2$, which corresponds to a restoring force of the form

$$F(x)=-dU/dx=-kx \tag{6}$$

The periodic motion is then *simple harmonic motion* (SHM) and the period (5) reduces to

$$P=2(m/k)^{1/2}\int_{-1}^{1}\left(1-u^2\right)^{-1/2} du\;=\;2(m/k)^{1/2}\left[\arcsin u\right]_{-1}^{1}$$

$$=2(m/k)^{1/2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]\;\Rightarrow$$

$$P=2\pi\left(\frac{m}{k}\right)^{1/2}\equiv\frac{2\pi}{\omega}\quad\text{where}\quad\omega=\frac{2\pi}{P}=\left(\frac{k}{m}\right)^{1/2}.$$

We notice that the period of SHM is amplitude-independent, hence also energy-independent.

It is of interest to examine a one-dimensional periodic motion that follows a *curved* path (where by "one-dimensional" we now mean that a single generalized coordinate – such as, e.g., an angle or a distance along the curve – is needed in order to specify the location of the particle). A nice example is that of an oscillating pendulum (Fig. 2; see also Sec. 5.5 and Problem 25 of [1]). The position of the mass m is specified by the arc length $OA=s=l\theta$ or, equivalently, by the angle θ (in rad). The algebraic value of the velocity of m is $v=ds/dt=ld\theta/dt$; it may be positive or negative, depending on the direction of motion relative to the unit tangent vector $\hat{u}_T$.

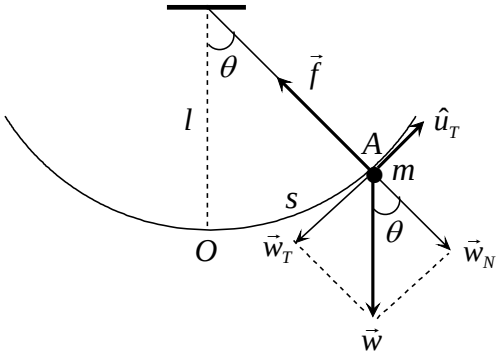

Fig. 2

The motion is governed by the tangential component $w_T = -mg\sin\theta$ (algebraic value) of the weight w. The tangential equation of motion of m is

$$m\,dv/dt = -mg\sin\theta \quad \Rightarrow \quad dv/dt = -g\sin\theta \tag{7}$$

We seek a conserved quantity that associates the velocity v with the position θ. We could, of course, work with (7) directly, but there is an easier way; namely, conservation of mechanical energy. This principle may be applied in view of the fact that the mass m is subject to the conservative force of gravity and the tension f of the string which, being normal to the velocity, produces no work (cf. Sec. 4.5 of [1]). The potential energy of m at point A (Fig. 2) is

$$U(\theta) = mg\,(l - l\cos\theta) = mgl\,(1-\cos\theta)\,,$$

where we have assumed that $U(0)=0$ (i.e., U is zero at the lowest point O). If α is the angular amplitude of oscillation (i.e., the maximum angle of deflection of the string from the vertical) then at $\theta=\pm\alpha$ the kinetic energy T vanishes and the total mechanical energy E is equal to $U(\pm\alpha)$. Applying conservation of mechanical energy between an arbitrary angle θ and the maximum angle $\theta=\alpha$, we have:

$$m\,v^2/2 + mgl\,(1-\cos\theta) = 0 + mgl\,(1-\cos\alpha) \;\Rightarrow\; \text{(after eliminating } m\text{)}$$

$$v^2 = 2gl\,(\cos\theta - \cos\alpha) \tag{8}$$

Exercise: By differentiating (8) with respect to t and by using the fact that $v=ld\theta/dt$, recover the equation of motion (7). Conversely, show that (8) is a direct consequence of (7). [*Hint:* Multiply (7) by v.]

Setting $v=ld\theta/dt$ in (8), we get a first-order differential equation:

$$d\theta/dt = \pm[(2g/l)(\cos\theta - \cos\alpha)]^{1/2},$$

which is integrated to give

$$\int_{\theta_0}^{\theta} \pm\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]^{-1/2} d\theta = t - t_0 .$$

The period of oscillation is [cf. Eq. (3)]

$$\begin{aligned} P &= 2\int_{-\alpha}^{\alpha}\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]^{-1/2} d\theta \\ &= (2l/g)^{1/2}\int_{-\alpha}^{\alpha}(\cos\theta - \cos\alpha)^{-1/2} d\theta \end{aligned} \tag{9}$$

Obviously, P depends on the angular amplitude α. Let us assume, however, that this amplitude is very small: $\alpha << 1$. We may then make the approximations

$$\cos\theta \approx 1 - \theta^2/2 \quad \text{and} \quad \cos\alpha \approx 1 - \alpha^2/2 .$$

Furthermore, we set $\theta/\alpha = u \Leftrightarrow \theta = \alpha u$. It is then a straightforward exercise to show that (9) reduces to

$$\begin{aligned} P &= 2(l/g)^{1/2}\int_{-1}^{1}\left(1-u^2\right)^{-1/2} du \;=\; 2(l/g)^{1/2}\left[\arcsin u\right]_{-1}^{1} \\ &= 2(l/g)^{1/2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] \;\Rightarrow \end{aligned}$$

$$P = 2\pi(l/g)^{1/2},$$

which is the familiar expression for the period of oscillation of a pendulum executing simple harmonic motion for small angles of deflection from the vertical. Once again, the SHM is seen to be the only one-dimensional periodic motion in which the period does not depend on the amplitude of oscillation.

Reference

[1] C. J. Papachristou, *Introduction to Mechanics of Particles and Systems* (Springer, 2020).[4]

[4] Manuscript: http://metapublishing.org/index.php/MP/catalog/book/68

Amplitude dependence of period in one-dimensional periodic motion

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Conservation of mechanical energy furnishes a neat way to formally evaluate the period of a one-dimensional periodic motion. It is shown that the only such motion where the period does not depend on the amplitude of oscillation – thus on the total energy of the oscillating body – is simple harmonic motion.

Consider a particle of mass m, moving along the x-axis under the action of a total force $F(x)$. The position $x(t)$ of the particle as a function of time is found by integrating the second-order differential equation (Newton's second law)

$$m\,d^2x/dt^2 = F(x) \tag{1}$$

for given initial conditions $x(t_0)=x_0$ and $v(t_0)=v_0$, where $v=dx/dt$ is the velocity of the particle.

Newton's law (1) may be rewritten as a system of first-order equations:

$$dx/dt = v\,, \qquad m\,dv/dt = F(x) \tag{2}$$

Dividing these equations in order to eliminate dt, we have:

$$m\,v\,dv = F(x)\,dx = -\,dU$$

where

$$U(x) = -\int_0^x F(x')dx' \quad \Leftrightarrow \quad F(x) = -\,dU/dx\,.$$

Thus, $m\,v\,dv + dU = d\,(m\,v^2/2 + U) = 0 \;\Rightarrow$

$$m\,v^2/2 + U(x) \equiv T + U = E = const. \tag{3}$$

(where T = kinetic energy) which expresses conservation of total mechanical energy.

From relation (3) we get

$$(dx/dt)^2 = (2/m)\,[E - U(x)] \;\Rightarrow\; dx/dt = \pm\,\{(2/m)\,[E - U(x)]\}^{1/2}\,.$$

Integrating this first-order differential equation and taking into account the initial condition $x=x_0$ for $t=t_0$, we have:

$$\int_{x_0}^{x} \frac{\pm\,dx}{\left\{\dfrac{2}{m}\big[E - U(x)\big]\right\}^{1/2}} = t - t_0 \tag{4}$$

where the plus sign is chosen for motion in the *positive* direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the *negative* direction ($v<0$, $x<x_0$). The value of the constant E in (4) may be determined by applying the initial conditions to (3): $E=mv_0^2/2+U(x_0)$, or by other physical considerations pertaining to the problem.

Let us now assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x\neq 0$ (this arrangement is always possible because of the arbitrariness in the definition of the zero-level of the potential energy). The graph of $U(x)$ is assumed to be symmetric with respect to the axis $x=0$, which means that $U(x)$ is an *even* function: $U(-x)=U(x)$.

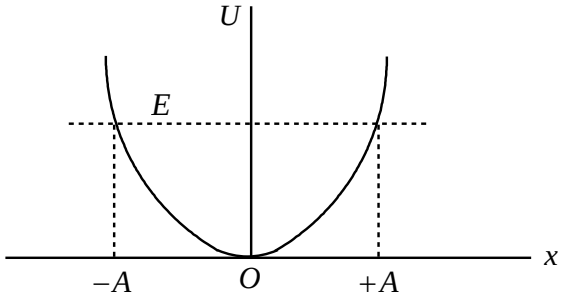

Fig. 1

Let E be the total mechanical energy of the particle. Since $E=T+U$ with $T\geq 0$, it follows that $E\geq U(x)$ for any physical motion. The motion is thus *bounded* between the points $-A$ and $+A$ of the x-axis (see Fig. 1), these points being *turning points* at which the particle stops momentarily ($E=U \Rightarrow T=0 \Rightarrow v=0$). Now, since E is constant, its value at all points equals its value at the turning points; i.e.,

$$E = U(\pm A) \tag{5}$$

The time it takes for a complete journey from $-A$ to $+A$ and back to $-A$ is found by using (4) with the appropriate sign for each direction of motion:

$$P = \int_{-A}^{A} \frac{dx}{\{\ldots\}^{1/2}} + \int_{A}^{-A} \frac{-dx}{\{\ldots\}^{1/2}} \;\Rightarrow$$

$$P = 2\int_{-A}^{A} \frac{dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = \left(2m\right)^{1/2} \int_{-A}^{A} \left[E-U(x)\right]^{-1/2} dx \tag{6}$$

[Since $E-U(x)$ is an even function, $\int_{-A}^{A} (E-U)^{-1/2} dx = 2\int_{0}^{A} (E-U)^{-1/2} dx$.]

Given that P is fixed for a given A, the motion is periodic about the point $x=0$, with amplitude equal to A and with period P. It follows from (6) that the period P depends on A and, therefore, on the total energy E of the particle, according to (5). We will now show that an exception where P does *not* depend on A (thus on E also) is *simple harmonic motion*.

Since $U(x)$ is an even function with $U(0)=0$, it can be expanded into a Maclaurin series of the form

$$U(x) = \sum_{l=1}^{\infty} a_l \, x^{2l} \tag{7}$$

where the coefficients a_l are not necessarily all different from zero. From (5) we have

$$E = U(\pm A) = \sum_{l=1}^{\infty} a_l \, A^{2l}$$

so that

$$E - U(x) = \sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right).$$

Equation (6) then yields

$$P = \left(2m\right)^{1/2} \int_{-A}^{A} \left[\sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right) \right]^{-1/2} dx \, .$$

By setting $x/A=u \;\Leftrightarrow\; x=Au$, we get:

$$P = \left(2m\right)^{1/2} A \int_{-1}^{1} \left[\sum_{l=1}^{\infty} a_l \, A^{2l} \left(1 - u^{2l} \right) \right]^{-1/2} du \tag{8}$$

It is obvious that, in general, P depends on A. The only exception where P is *not* dependent on A is the case where the following condition is satisfied: $a_l = 0$ for $l \neq 1$. That is, the only nonvanishing coefficient a_l in the series (7) is a_1 . By setting $a_1 = k/2$ the potential energy (7) reduces to $U(x) = kx^2/2$, which corresponds to a restoring force of the form

$$F(x) = - \, dU / dx = - kx \tag{9}$$

The periodic motion is then *simple harmonic motion* (SHM) and the period (8) reduces to

$$P = 2(m/k)^{1/2} \int_{-1}^{1} \left(1 - u^2\right)^{-1/2} du \; = \; 2(m/k)^{1/2} \left[\arcsin u \right]_{-1}^{1}$$

$$= 2(m/k)^{1/2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] \;\Rightarrow$$

$$P = 2\pi \left(\frac{m}{k} \right)^{1/2} \equiv \frac{2\pi}{\omega} \quad \text{where} \quad \omega = \frac{2\pi}{P} = \left(\frac{k}{m} \right)^{1/2} .$$

We notice that the period of SHM is *amplitude-independent*, hence also *energy-independent*.

We may obtain the equation of motion $x=x(t)$ for SHM by using (4) with $U(x)=kx^2/2$ and $E=U(\pm A)=kA^2/2$. Let us assume first that the motion is in the positive direction, so that $x>x_0$. Setting $\omega=(k/m)^{1/2}$, we have:

$$\int_{x_0}^{x}\left(A^2-x^2\right)^{-1/2}dx=\omega\,(t-t_0)\,.$$

Using the integral formula

$$\int\left(A^2-x^2\right)^{-1/2}dx=\arcsin(x/A)+C$$

and making appropriate substitutions for constants, we find an equation of the form[1]

$$\arcsin(x/A)=\omega t+\alpha \quad\Rightarrow\quad x=A\sin(\omega t+\alpha)\,.$$

For motion in the negative direction ($x<x_0$) we choose the minus sign in (4), so that

$$\int_{x_0}^{x}\left(A^2-x^2\right)^{-1/2}dx=-\omega\,(t-t_0)\,.$$

This yields a result of the form[2]

$$\arcsin(x/A)=-\omega t+\beta \quad\Rightarrow\quad x=-A\sin(\omega t-\beta)\,.$$

Since the constant β is arbitrary (being dependent on the arbitrary constants x_0 and t_0) we may set $-\beta\equiv\pi+\alpha$, so that $x=A\sin(\omega t+\alpha)$, as before.

We conclude that the general solution of the differential equation (1) for SHM under the action of a force (9), is

$$x(t)=A\sin(\omega t+\alpha)\,.$$

Physically, A is the *amplitude* of oscillation, $\omega=(k/m)^{1/2}$ is the *angular frequency* and α is the *initial phase* (i.e., the *phase* $\omega t+\alpha$ at $t=0$).

It is of interest to examine a one-dimensional periodic motion that follows a *curved* path (where by "one-dimensional" we now mean that a single generalized coordinate – such as, e.g., an angle or a distance along the curve – is needed in order to specify the location of the particle). A nice example is that of an oscillating pendulum, shown in Fig. 2 (see also [1-3]). The position of the mass m is specified by the arc length $OA=s=l\theta$ or, equivalently, by the angle θ (in *rad*). The algebraic value of the velocity of m is $v=ds/dt=ld\theta/dt$; it may be positive or negative, depending on the direction of motion relative to the unit tangent vector $\hat{u}_T$.

[1] Explicitly: $\alpha=\arcsin(x_0/A)-\omega t_0$.
[2] Explicitly: $\beta=\arcsin(x_0/A)+\omega t_0$.

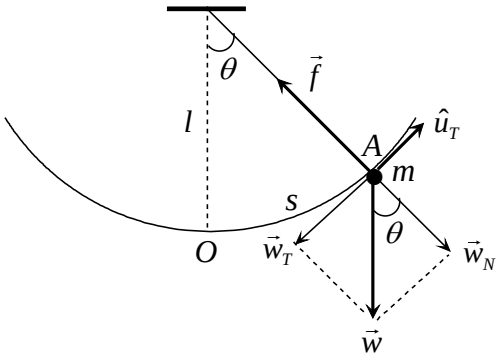

Fig. 2

The motion is governed by the tangential component $w_T = -mg\sin\theta$ (algebraic value) of the weight w. The tangential equation of motion of m is

$$m\,dv/dt = -mg\sin\theta \quad \Rightarrow \quad dv/dt = -g\sin\theta \tag{10}$$

We seek a conserved quantity that associates the velocity v with the position θ, in the spirit of Eq. (3). We could, of course, work with (10) directly but there is an easier way; namely, conservation of mechanical energy. This principle may be applied in view of the fact that the mass m is subject to the conservative force of gravity and the tension f of the string which, being normal to the velocity, produces no work (cf. Sec. 4.5 of [1]). The potential energy of m at point A (Fig. 2) is

$$U(\theta) = mg\,(l - l\cos\theta) = mgl\,(1-\cos\theta)\,,$$

where we have assumed that $U(0)=0$ (i.e., U is zero at the lowest point O). If α is the angular amplitude of oscillation (i.e., the maximum angle of deflection of the string from the vertical) then at $\theta = \pm\alpha$ the kinetic energy T vanishes and the total mechanical energy E is equal to $U(\pm\alpha)$. Applying conservation of mechanical energy between an arbitrary angle θ and the maximum angle $\theta=\alpha$, we have:

$$m\,v^2/2 + mgl\,(1-\cos\theta) = 0 + mgl\,(1-\cos\alpha) \;\Rightarrow\; \text{(after eliminating } m\text{)}$$

$$v^2 = 2gl\,(\cos\theta - \cos\alpha) \tag{11}$$

Exercise: By differentiating (11) with respect to t and by using the fact that $v=ld\theta/dt$, recover the equation of motion (10). Conversely, show that (11) is a direct consequence of (10). [*Hint:* Multiply (10) by v.]

Setting $v=ld\theta/dt$ in (11), we get a first-order differential equation:

$$d\theta/dt = \pm\left[(2g/l)(\cos\theta - \cos\alpha)\right]^{1/2},$$

which is integrated to give

$$\int_{\theta_0}^{\theta} \pm\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]^{-1/2} d\theta = t - t_0\ .$$

The period of oscillation is [cf. Eq. (6)]

$$\begin{aligned} P &= 2\int_{-\alpha}^{\alpha}\left[\frac{2g}{l}(\cos\theta-\cos\alpha)\right]^{-1/2}d\theta \\ &= (2l/g)^{1/2}\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)^{-1/2}d\theta \end{aligned} \tag{12}$$

Obviously, P depends on the angular amplitude α. Let us assume, however, that this amplitude is very small: $\alpha<<1$. We may then make the approximations

$$\cos\theta \approx 1-\theta^2/2 \quad \text{and} \quad \cos\alpha \approx 1-\alpha^2/2\,.$$

Furthermore, we set $\theta/\alpha=u \Leftrightarrow \theta=\alpha u$. It is then a straightforward exercise to show that (12) reduces to

$$\begin{aligned} P &= 2(l/g)^{1/2}\int_{-1}^{1}\left(1-u^2\right)^{-1/2}du \;=\; 2(l/g)^{1/2}\left[\arcsin u\right]_{-1}^{1} \\ &= 2(l/g)^{1/2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] \;\Rightarrow \end{aligned}$$

$$P = 2\pi(l/g)^{1/2}\,,$$

which is the familiar expression for the period of oscillation of a pendulum executing simple harmonic motion for small angles of deflection from the vertical. Once again, the SHM is seen to be the only one-dimensional periodic motion in which the period does not depend on the amplitude of oscillation.

As another example, consider a body of mass m, which is moving back and forth on a U-shaped, frictionless roller-coaster track on the vertical xy-plane, where the x-axis is horizontal while the y-axis is vertical (Fig. 3). The shape of the track, which is symmetric with respect to the y-axis, is described mathematically by an equation of the form $y=f(x)$, where $f(x)$ is an *even* function and where $f(0)=0$. We want to determine the period of the oscillatory motion, given the total mechanical energy E of m (equivalently, the maximum height h reached by the body).

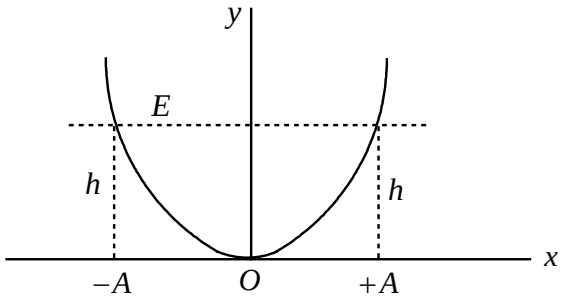

Fig. 3

Let us first take a look at the physics of the problem. The body m is sliding without friction on the roller-coaster track, moving back and forth between two extreme points at height h above the x-axis (Fig. 3). The projections of these points on this axis are $-A$ and $+A$. The body is subject to the gravitational force mg and the

normal force from the track. The latter force produces no work, hence does not affect the conservation of mechanical energy (see Sec. 4.5 of [1]). The gravitational potential energy of m is $U(y)=mgy$. Along the track, where $y=f(x)$, the values of U may be expressed in terms of x:

$$U(x) = mg\,f(x) \tag{13}$$

Let E be the total mechanical energy of m. Since E is constant along the path, its value will be equal to the value of the potential energy at the extreme positions corresponding to $x= -A$ and $x=+A$ (at which positions the kinetic energy of m vanishes). That is,

$$E = U(\pm A) = mg\,f(\pm A) = mgh \tag{14}$$

The kinetic energy of the body is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

(dots indicate differentiation with respect to t) where, for $y=f(x)$,

$$\dot{y} = \frac{d}{dt}\,f(x) = \frac{df(x)}{dx}\frac{dx}{dt} = \dot{x}f'(x) \tag{15}$$

Hence,

$$T = \frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} \tag{16}$$

The total mechanical energy $E=T+U$ is constant along the path. By (13), (14) and (16) we have:

$$\frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} + mg\,f(x) = mgh \tag{17}$$

The position of m on the track is specified by a single coordinate x, which plays the role of a generalized coordinate in the sense of Lagrangian dynamics. The Lagrangian function is

$$L(x,\dot{x}) = T - U = \frac{1}{2}m\dot{x}^2\left\{1+[f'(x)]^2\right\} - mg\,f(x) \tag{18}$$

The Lagrange equation for $x(t)$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{19}$$

We note that the time-derivative of *any* function of x is defined by the rule used in (15) for $f(x)$. With this in mind, it is a somewhat long but straightforward exercise to show that (18) and (19) yield the differential equation

$$\ddot{x}\left\{1+[f'(x)]^2\right\}+\dot{x}^2 f'(x) f''(x)+g f'(x)=0 \tag{20}$$

Presumably, the first-order differential equation (17) for x, expressing conservation of mechanical energy, is a *first integral* of the second-order differential equation (20). (In general, a first integral of a differential equation is a lower-order differential equation – or an algebraic relation, in the case of a first-order equation – that gives us the information that some mathematical quantity retains a constant value as a consequence of the original differential equation; see, e.g., [4].) To prove the validity of the above statement, we need to integrate (20) once with respect to t in order to derive (17). It is easier, however, to work in reverse order. We thus take the time-derivative of (17), keeping the rule (15) in mind. Not surprisingly, the result is again the differential equation (20) (show this)!

The equation of motion of m on the track is a function $x(t)$ that satisfies the differential equation (20). In principle, this second-order equation has "already" been integrated once to obtain the first-order equation (17) [which is a first integral of (20), expressing conservation of mechanical energy]. From (17) we have:

$$\dot{x}^2=\frac{2g[h-f(x)]}{1+[f'(x)]^2}\ .$$

This yields a first-order differential equation for $x(t)$:

$$\frac{dx}{dt}=\pm\left\{\frac{2g[h-f(x)]}{1+[f'(x)]^2}\right\}^{1/2}\equiv\pm\Lambda(x;h) \tag{21}$$

By assuming the initial condition $x=x_0$ for $t=t_0$, the differential equation (21) is integrated to give

$$\int_{x_0}^{x}\frac{\pm\,dx}{\Lambda(x;h)}=t-t_0 \tag{22}$$

where the plus sign is chosen for motion in the positive direction ($x>x_0$), while the minus sign applies to motion in the negative direction ($x<x_0$). This formally solves the problem of determining the position of m on the track as a function of time.

The period P of the oscillatory motion of m is the time it takes for a complete journey from the extreme position with $x= -A$ to the extreme position with $x= +A$ and back to the original position $x= -A$. To find P we use (22) with the appropriate sign for each direction of motion:

$$P=\int_{-A}^{A}\frac{dx}{\Lambda(x;h)}+\int_{A}^{-A}\frac{-dx}{\Lambda(x;h)}=2\int_{-A}^{A}\frac{dx}{\Lambda(x;h)}\ .$$

We observe that P depends on the maximum height h, thus on the total energy E of the body (notice that both the integrand *and* the limits of integration depend on h). However, P is independent of the mass of the body, as expected for a motion governed by the sole action of gravity.

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2. J. B. Marion, S. T. Thornton, *Classical Dynamics of Particles and Systems*, 4th Edition (Saunders College, 1995).
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[3] See http://metapublishing.org/index.php/MP/catalog/book/68

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Revisiting Archimedes' principle: Buoyancy and external pressure

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A careful examination of Archimedes' principle shows that the buoyant force on a body that is either fully or partially immersed in a liquid is unaffected by the external (e.g. atmospheric) pressure, which acts both on the non-immersed part of the body (if any) and on the immersed part via Pascal's principle. The net force on the body due to the external pressure is zero and hence this pressure does not contribute to buoyancy.

1. A proper understanding of the buoyant force

In the statement of Archimedes' principle [1,2] the buoyant force on a body that is fully or partially immersed in a liquid is defined as the resultant of all elementary normal forces exerted by the liquid on the immersed surface of the body. It is then stated that this upward force is equal in magnitude to the weight of the fluid displaced by the immersed part of the body. This definition of the buoyant force turns out to be consistent with Archimedes' principle for a body that is *fully* immersed. The situation is subtler, however, in the case of a *partially* immersed floating body.

The force exerted by the fluid on the immersed surface S of the body is due to the total pressure P at the various points of S. According to Pascal's principle, this pressure is equal to the sum $P=P_l+P_0$ of the hydrostatic pressure P_l due to the liquid itself and the *constant* external pressure P_0. The buoyant force on the body is typically defined as the total force on S due to P.

In the case of a fully immersed body the immersed surface S, as well as the surface of the displaced fluid, coincides with the total surface of the body. Moreover, as will be shown, the external pressure P_0 does not contribute to the total force on any closed surface. The buoyant force is thus exclusively due to the hydrostatic pressure P_l of the liquid itself, regardless of the value of the external pressure. By the equilibrium condition for the displaced fluid, the weight of the latter is equal in magnitude and opposite in direction relative to the buoyant force (see Appendix).

In the case of a partially immersed floating body the immersed surface S is only a part of the total surface of the body. Likewise, the surface S constitutes only a part of the total surface of the displaced liquid. The non-immersed surface of the body, as well as the top surface of the displaced liquid, is subject only to the external pressure P_0. Now, what is typically called "buoyant force" in this case is the total force on the immersed surface S of the body, which force is due to the total pressure $P=P_l+P_0$ at each point of S. By the equilibrium condition this force is assumed to be equal in magnitude to the weight of the body. But such a "balance" of forces makes no sense, given that the weight is a fixed force while the force on the immersed surface S may vary arbitrarily by changing the external pressure P_0 (this pressure is transferred to all points of S and adds to P_l in accordance with Pascal's principle). To restore the balance of forces we must include the *downward* force on the *non-immersed* surface of the body due to the external pressure. As it turns out, this force exactly matches the *upward* Pascal-oriented force on the immersed surface S due to P_0 alone, so that, eventually, the force exerted over the *entire* surface of the body (both immersed and

non-immersed) by the external pressure is zero. All we are left with, therefore, is the hydrostatic force on S due to the pressure P_l of the liquid alone. It is *this* force that will properly balance the weight of the body. Also, it is *this* force that will balance the weight of the displaced liquid. It is thus clear that, for Archimedes' principle to be satisfied, it is the force due to P_l (*not* the force due to the total pressure P) that must be identified as the buoyant force in the case of a *floating* body. For a *fully immersed* body, where the immersed surface S is the total surface of the body, the total force due to P reduces to that due to P_l ; it is thus permissible to define the former force as the buoyant force in this case.

In conclusion: For consistency with Archimedes' principle regardless of whether a body is fully or partially immersed in a liquid, we must generally define the buoyant force as the total force on the immersed surface S of the body due to the pressure P_l exerted *by the liquid alone*. Moreover, as shown below, the (constant) external pressure P_0 contributes no additional net force on the body as a whole.

By properly defining the buoyant force, the balance of forces for a floating body, expressed by the equilibrium condition "buoyant force = total weight of the body", determines the percentage of the total volume of the body that is immersed in the liquid (cf. Sec. 8.9 of [1]). Since, as said above, the total force on the body is independent of the external pressure, it follows that we cannot make a floating body immerse further by increasing this pressure!

2. Constant external pressure on a closed surface

We propose to show that a *constant* external pressure P_0 does not affect the total force on a body that is either fully or partially immersed in a liquid. [This pressure is felt directly on the non-immersed part (if any) as well as on the immersed part via Pascal's principle.] This means that an additional constant pressure over the *entire* surface of the body does not change the total force that would be exerted on the body by the liquid alone (i.e., if the external pressure P_0 did not exist). The force on the entire surface of the body due to a constant external pressure P_0 must thus be zero.

Proposition: Consider a closed surface S inside a scalar field of constant value P_0 (Fig. 1). At each infinitesimal element ds of S the field exerts a force $d\vec{F}$ normal to ds and having magnitude dF proportional to the area of this surface element (which area will also be called ds): $dF=P_0\,ds$. We assume that, at each point of S, the elementary normal force $d\vec{F}$ on the local surface element ds is directed *toward* the surface, i.e., opposite to the local unit vector $\hat{n}$ that is normal to S and directed *outward*. Then, the total force exerted on S by the field P_0 is zero.

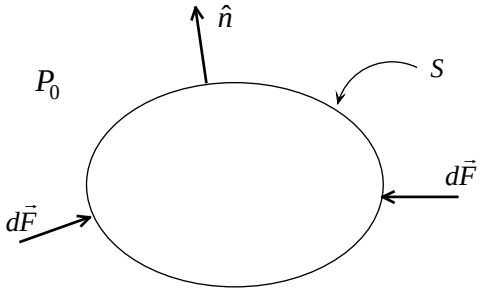

Fig. 1

Proof: We have that $d\vec{F} = -dF\,\hat{n} = -P_0\,ds\,\hat{n}$, so that the total force on *S* is

$$\vec{F} = \oint_S d\vec{F} = -P_0 \oint_S \hat{n}\,ds \tag{1}$$

We show that, for any closed surface *S*, the following integral relation is true:

$$\vec{I} \equiv \oint_S \hat{n}\,ds = 0 \tag{2}$$

It suffices to show that this vector relation is true when projected to *any* arbitrary direction. Let $\hat{b}$ be a unit vector defining such a direction. We write

$$I_b = \vec{I} \cdot \hat{b} = \oint_S \hat{b} \cdot \hat{n}\,ds\,.$$

Now, consider the constant vector field $\vec{f}(\vec{r}) = \hat{b}$. By using Gauss' integral theorem [3] we have:

$$I_b = \oint_S \vec{f}(\vec{r}) \cdot \hat{n}\,ds = \int_V (\vec{\nabla} \cdot \vec{f})\,dV = 0 \quad [\text{since } \vec{\nabla} \cdot \vec{f}(\vec{r}) = \vec{\nabla} \cdot \hat{b} = 0\,]$$

where *V* is the volume enclosed by the surface *S*. Thus, the projection of the vector-valued integral $\vec{I} \equiv \oint_S \hat{n}\,ds$ to *any* arbitrary direction vanishes, which means that the vector relation (2) is true. Accordingly, the total force $\vec{F}$ on *S*, given by Eq. (1), is zero.

An alternative, more "intuitive" proof of the above Proposition is the following: Since *S* is a closed surface, for any unit vector $\hat{n}$ normal to *S* at some point of this surface there exists another point of *S* at which the normal unit vector is directed opposite to $\hat{n}$ (of course, both unit vectors are directed *outward* relative to the surface). This is easier to understand if instead of a closed surface we consider a closed plane curve *C* (see Fig. 2). If we make a full trip on *C*, the normal unit vector $\hat{n}$ will assume all possible directions until it finally returns to its original direction at the starting point of the trip. One of these (infinitely many) directions will be the opposite of the initial direction of $\hat{n}$.

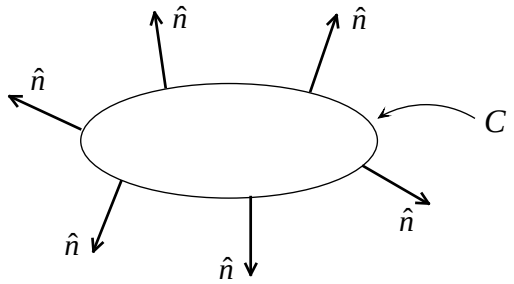

Fig. 2

Going back to our closed surface *S*, it follows from the above discussion that for every surface element $\hat{n}\,ds$ there is a corresponding element with opposite direction. This implies that $\oint_S \hat{n}\,ds = 0$ (which is an interesting mathematical result in its own right). Hence, by Eq. (1), the total force $\vec{F}$ on *S* is zero. As seen in Fig. 3, for every elementary force $d\vec{F}$ on *S* there is always an opposite force $-d\vec{F}$ acting at some other point of the surface, so that, eventually, the net force on *S* by the constant field P_0 is zero.

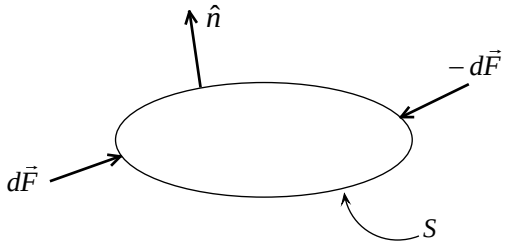

Fig. 3

In conclusion: A constant external pressure P_0 has no effect on the total force experienced by a body that is either totally or partially immersed in a liquid. In particular, the equilibrium situation of a floating body will not be altered if we increase or decrease the external pressure.

Appendix: Proof of Archimedes' principle for a fully immersed body

For a fully immersed body the principle is proven theoretically as follows: Let us call V_d and $\vec{W}_d$ the volume and the weight, respectively, of the fluid displaced by the body. Since the body is fully immersed in the liquid, V_d equals the volume of the body.

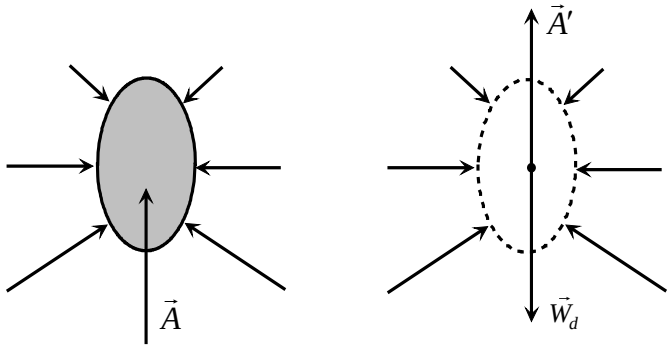

Fig. 4

Part (*a*) of Fig. 4 shows an instantaneous picture of the immersed body. The word "instantaneous" is related to the fact that, in general, the body is *not* in a state of equilibrium inside the liquid. The buoyant force $\vec{A}$ is typically defined as the resultant of all elementary forces acting normally on the surface of the body by the liquid.

In part (*b*) of Fig. 4 the body has been removed and has been replaced by liquid of the same volume and shape. The surface of that section of the fluid is now subject to a total force $\vec{A}'$ (buoyant force) from the surrounding fluid. The weight $\vec{W}_d$ of this fluid section is equal to the weight of the fluid that had previously been displaced by the body, while the line of action of $\vec{W}_d$ passes through the center of gravity of the displaced fluid.

In contrast to the submerged body, the part of the liquid that replaced the body is in a state of equilibrium since it is a portion of a fluid at rest. Hence,

$$\vec{A}' + \vec{W}_d = 0 \quad \Rightarrow \quad \vec{A}' = -\vec{W}_d \ .$$

Now, the buoyant force on the body is the same as the buoyant force on the part of the fluid replacing the body (i.e., $\vec{A} = \vec{A}'$) since the elementary forces exerted by a fluid on a surface are independent of the nature of the surface [1]. Thus, finally, the buoyant force exerted by the fluid on the body is $\vec{A} = -\vec{W}_d$. The direction of $\vec{A}$ is upward (i.e., opposite to the direction of $\vec{W}_d$) while its magnitude is $A = W_d = \rho g V_d$, where ρ is the density of the liquid.

We note that the total force on the surface of the fully immersed body contains contributions from the constant external pressure P_0, which pressure is transferred via Pascal's principle to all points of the liquid. As we have shown, however, the net force due to P_0 over any closed surface (hence the surface of the body) is zero. Thus the buoyant force $\vec{A}$, which was defined as the total force exerted by the surrounding liquid, is eventually *independent* of the external pressure P_0 and equal to the force due to the pressure P_l of the liquid itself.

The case of a *partly* immersed floating body is subtler, as we discussed earlier. Consistency with Archimedes' principle suggests that the properly defined buoyant force is the force due to the pressure P_l exerted on the immersed part of the body by the liquid alone, while the external pressure P_0 (acting on both immersed and non-immersed parts of the body) contributes no extra net force on the body as a whole. Thus the buoyant force is independent of external pressure and equal in magnitude to the weight of the displaced fluid, in accordance with Archimedes' principle.

References

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