Διανύσματα στον χώρο

$P(x_1, y_1, z_1) \quad Q(x_2, y_2, z_2)$

$$\vec{PQ} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

$$|\vec{PQ}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Εσωτερικό γινόμενο διανυσμάτων

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

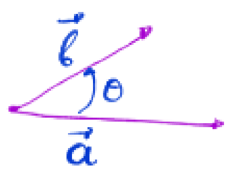
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Ιδιότητες

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot (\vec{b} + \vec{\gamma}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{\gamma}$
3. $\lambda (\vec{a} \cdot \vec{b}) = (\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b})$
4. $\vec{a} \cdot \vec{a} = \vec{a}^{\,2} = |\vec{a}|^2$

Ισοδύναμος ορισμός εσωτερικού γινομένου

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sigma\upsilon\nu\theta$$



$$\sigma\upsilon\nu\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

- $\vec{a} \perp \vec{b} \leftrightarrow \vec{a} \cdot \vec{b} = 0$

Άσκηση

Να βρεθεί το λ ώστε τα $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ κ' $\vec{b} = \vec{i} + \lambda \vec{j} + \vec{k}$

i) να είναι κάθετα

iii) να σχηματίζουν γωνία $\frac{\pi}{3}$

Λύση i) $\vec{a}\cdot\vec{b}=0 \Rightarrow (1,2,-1)\cdot(1,\lambda,1)=0 \Rightarrow$

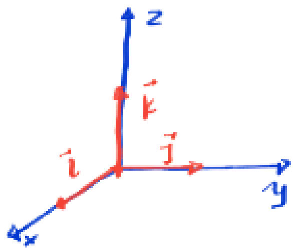
$1+2\lambda-1=0 \Rightarrow \boxed{\lambda=0}$

ii) $\sigma\upsilon\nu\frac{\pi}{3}=\frac{2\lambda}{\sqrt{6}\cdot\sqrt{2+\lambda^2}} \Rightarrow \frac{1}{2}=\frac{2\lambda}{\sqrt{6(2+\lambda^2)}}$

$\Rightarrow 4\lambda=\sqrt{12+6\lambda^2} \Rightarrow 16\lambda^2=12+6\lambda^2$

$\Rightarrow \lambda=\pm\sqrt{\frac{6}{5}}$

Εξωτερικό γινόμενο διανυσμάτων

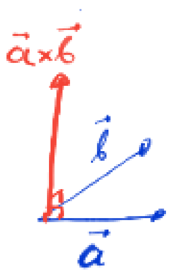


$\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=(a_2b_3-b_2a_3,\,-a_1b_3+b_1a_3,\,a_1b_2-b_1a_2)$$

Γεωμετρική ερμηνεία

$\vec{a}\times\vec{b}\perp\vec{a}$



$\vec{a}\times\vec{b}\perp\vec{b}$

Ιδιότητες

1. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$
2. $\vec{a}\times(\vec{b}+\vec{\gamma})=\vec{a}\times\vec{b}+\vec{a}\times\vec{\gamma}$
3. $\lambda\cdot(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

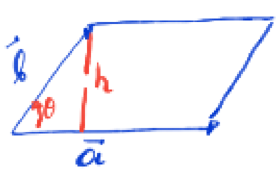
4. $a \times b = 0 \Rightarrow a, b$ συγγραμμικά

• Ισχύει ότι $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot ημθ$

Εμβαδόν παραλληλογράμμου

$$Ε = |\vec{a}| \cdot h = |\vec{a}| \cdot |\vec{b}|\, ημθ$$

$$\Rightarrow Ε = |\vec{a} \times \vec{b}|$$



$h = |\vec{b}|\, ημθ$

Άσκηση

Αν $\vec{a} + \vec{b} + \vec{γ} = \vec{0}$. Να δειχθεί ότι $\vec{a} \times \vec{b} = \vec{b} \times \vec{γ} = \vec{γ} \times \vec{a}$

Λύση

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{γ}) = \vec{a} \times \vec{0}$$

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{γ} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{γ} \Rightarrow \vec{a} \times \vec{b} = \vec{γ} \times \vec{a} \quad (1)$$

$$\vec{γ} \times (\vec{a} + \vec{b} + \vec{γ}) = \vec{γ} \times \vec{0}$$

$$\vec{γ} \times \vec{a} + \vec{γ} \times \vec{b} + \vec{γ} \times \vec{γ} = \vec{0}$$

$$\vec{γ} \times \vec{a} = -\vec{γ} \times \vec{b} \Rightarrow \vec{γ} \times \vec{a} = \vec{b} \times \vec{a} \quad (2)$$

Μικτό γινόμενο διανυσμάτων

$\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{γ} = (γ_1, γ_2, γ_3)$

$$\vec{a} \cdot (\vec{b} \times \vec{γ}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ γ_1 & γ_2 & γ_3 \end{vmatrix} = (\vec{a}\ \vec{b}\ \vec{γ})$$

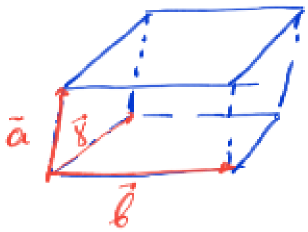
Ιδιότητες

Ιδιότητες

1. $(\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}) = -(\vec{\beta}\ \vec{\alpha}\ \vec{\gamma}) = (\vec{\beta}\ \vec{\gamma}\ \vec{\alpha})$
2. $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}+\vec{\delta}) = (\vec{\alpha}, \vec{\beta}, \vec{\gamma}) + (\vec{\alpha}, \vec{\beta}, \vec{\delta})$
3. $(\vec{\alpha}, \vec{\alpha}, \vec{\beta}) = 0$
4. $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = 0 \Leftrightarrow$ τα $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ είναι συνεπίπεδα

Όγκος παραλληλεπιπέδου

$$V = |(\vec{\alpha}, \vec{\beta}, \vec{\gamma})|$$



Άσκηση

$|\vec{\alpha}| = |\vec{\beta}| = |\vec{\gamma}| = |\vec{\delta}| = 1$

$(\widehat{\vec{\alpha}, \vec{\beta}}) = \frac{\pi}{4}$

$\vec{\gamma} \perp \vec{\alpha}$, $\vec{\gamma} \perp \vec{\beta}$

$\vec{\delta} \perp \vec{\alpha}$, $(\vec{\delta}, \vec{\beta}) = \frac{\pi}{3}$

Να βρεθούν τα λ, μ, ν ώστε $\vec{\delta} = \lambda\vec{\alpha} + \mu\vec{\beta} + \nu\vec{\gamma}$

Λύση

$$\vec{\alpha}\cdot\vec{\delta} = \lambda|\vec{\alpha}|^2 + \mu\,\vec{\alpha}\cdot\vec{\beta} + \nu\,\vec{\alpha}\cdot\vec{\gamma}$$

$$0 = \lambda + \mu|\vec{\alpha}|\cdot|\vec{\beta}|\cdot\sigma\upsilon\nu\frac{\pi}{4} \Rightarrow \quad 0 = \lambda + \frac{\sqrt{2}}{2}\mu \quad (1)$$

$$\vec{\beta}\cdot\vec{\delta} = \lambda\,\vec{\beta}\cdot\vec{\alpha} + \mu|\vec{\beta}|^2 + \nu\,\vec{\beta}\cdot\vec{\gamma}$$

$$\frac{1}{2} = \frac{\sqrt{2}}{2}\lambda + \mu \quad (2)$$

$$(1)+(2) \Rightarrow \quad \mu = 1, \quad \lambda = -\frac{\sqrt{2}}{2}$$

$$\vec{\delta}\cdot\vec{\delta} = -\sqrt{2}\,\vec{\delta}\cdot\vec{\alpha} + \vec{\delta}\cdot\vec{\beta} + \nu\,\vec{\delta}\cdot\vec{\gamma}$$

$$1 = \frac{1}{2} + \nu\, \vec{\delta} \cdot \vec{\gamma} \Rightarrow \vec{\delta} \cdot \vec{\gamma} = \frac{1}{2\nu}$$

$$\vec{\gamma} \cdot \vec{\delta} = -\frac{\sqrt{2}}{2}\, \vec{\gamma} \cdot \vec{a} + \vec{\gamma} \cdot \vec{b} + \nu (\vec{\gamma})^2$$

$$\frac{1}{2\nu} = \nu \Rightarrow \nu^2 = \frac{1}{2} \Rightarrow \nu = \pm \sqrt{\frac{1}{2}}$$