Motion of a charged particle in a uniform magnetic field

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Several aspects of the motion of a charged particle in a uniform magnetic field are examined, both by physical arguments and by explicit solution of the differential equation of motion.

Problem

A particle of mass m and charge $q>0$ enters a uniform magnetic field $\vec{B}$ with initial velocity $\vec{v}_0$ perpendicular to the field. The magnetic field is assumed to be oriented in the positive z-direction.

1. Show that the particle will execute uniform circular motion on the xy-plane and determine the radius r of this motion.

2. Show that the larger the momentum of the particle, the smaller the curvature of the path. Interpret this physically.

3. Determine the angular velocity ω of the particle and show that the period of revolution is independent of the size of the orbit.

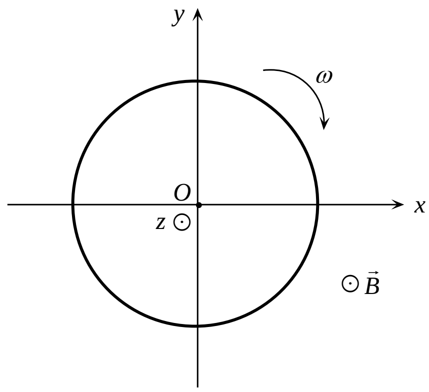
4. Suppose that the magnitude B of the magnetic field increases with time, although the field remains uniform (i.e., *spatially* constant) at all times. Show that the increase of B produces a decrease of the size of the orbit.

5. Assume now that the particle enters the magnetic field in a direction that is *not* perpendicular to the field. Show that the motion of the particle will be uniform, while the projection of this motion onto the xy-plane will be uniform circular with angular velocity ω equal to that found in part 3. Describe the path geometrically.

6. Show that the radiation losses due to acceleration become more significant the smaller the mass of the particle.

7. By solving the differential equation of motion of the charged particle, derive explicit expressions for the coordinates (x, y, z) of the particle as functions of time t. Demonstrate that the projection of the motion onto the xy-plane is uniform circular, as found previously, and verify the expression for the angular velocity ω. Explain why this planar motion is clockwise for the given direction of $\vec{B}$.

Solution



Both the *z*-axis and the magnetic field are normal to the page and directed toward the reader; the direction of motion is clockwise (why?).

1. The charged particle is subject to a magnetic force

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{1}$$

where, in components, $\vec{v} = v_x \hat{u}_x + v_y \hat{u}_y + v_z \hat{u}_z$ and $\vec{B} = B\hat{u}_z$ $(B = |\vec{B}|)$. Then,

$$\vec{F} = qB(v_y \hat{u}_x - v_x \hat{u}_y) \tag{2}$$

which is a vector in the *xy*-plane; the same is true, therefore, with regard to the acceleration of the particle (assuming no other forces act on it). Given that, by assumption, the initial velocity also is a vector in the *xy*-plane, we conclude that the motion of the particle takes place on that plane.

As seen in (1), the total force on the particle is normal to the particle's velocity, i.e., normal to the trajectory of the particle. This means that the particle moves at *constant speed* inside the magnetic field (see, e.g., Section 2.4 of [1] and Sec. 7.1 of [2]). In other words, the particle executes *uniform* curvilinear motion. We must now show that this motion is *circular*. Indeed, the magnitude of the magnetic force is

$$F = qvB = constant \tag{3}$$

where v is the (constant) speed of the particle, equal to the initial speed v_0, and where we have taken into account that the velocity vector is always perpendicular to the magnetic field. Now, since the motion is uniform, the total force (1) is purely centripetal. Hence, $F=mv^2/\rho$, where ρ is the radius of curvature at any point of the trajectory (see Sec. 3.6 of [1]). Given that both v and F are constant, it follows that ρ is constant also; that is, the motion is circular. We may place the center of the circle at the origin O of our coordinate system (in particular, of the *xy*-plane) so that the radius ρ of the circle equals the distance r of the particle from O. From $F=mv^2/r$, and by using (3), we find:

$$r = \frac{mv}{qB} \tag{4}$$

2. Let $p=mv$ be the (constant) magnitude of the momentum of the particle. Relation (4) may then be rewritten as $r=p/qB$. We observe that r is an increasing function of p: the larger the momentum, the larger the radius and, therefore, the smaller the curvature of the path. Physically, this means that as the momentum increases it becomes more difficult for the magnetic field to produce a change in the direction of motion of the particle.

3. We write $v=\omega r$, where ω is the angular velocity. Substituting this into (4), we find

$$\omega = \frac{qB}{m} \tag{5}$$

We notice that ω is independent of the radius r of the orbit; so is, therefore, the period $T=2\pi/\omega$ of the circular motion.

4. Since $v=v_0$=constant, independent of the magnetic-field strength B, a change of B will not affect the speed of the particle. From (4) it then follows that an increase of B will produce a decrease of r, i.e., of the size of the orbit. This means that the particle will revolve closer to the z-axis. This effect is used in fusion reactors to achieve plasma heating and confinement.

5. As argued in part 1 of the problem, since the total force on the particle is normal to the particle's velocity, the speed v of the particle is constant, equal to the initial speed v_0 , and the motion is *uniform curvilinear*. Furthermore, the total force, given by (1) and (2), is a vector parallel to the xy-plane, and so is the acceleration of the particle. These results are independent of the direction of the initial velocity of the charge upon its entrance into the magnetic field. Notice also that Eq. (2) is valid even if the velocity has a z-component.

The motion, however, is no longer expected to be planar if the direction of the velocity has a z-component, as will now be assumed to be the case. Let us write

$$\vec{v} = \vec{v}\,' + v_z \hat{u}_z \quad \text{where} \quad \vec{v}\,' = v_x \hat{u}_x + v_y \hat{u}_y \equiv \text{vector parallel to the } xy\text{-plane} \tag{6}$$

Since the z-component of the acceleration is zero, the velocity does not change in the z-direction; that is, $v_z=v_{0z}=$*constant*. Hence, along the z-axis (which is parallel to the magnetic field) the motion is uniform rectilinear. Regarding the motion parallel to the xy-plane, we note the following:

$$\vec{F} \cdot \vec{v}\,' = \vec{F} \cdot \left(\vec{v} - v_z \hat{u}_z \right) = 0\,,$$

since by (1) the total force is normal to the velocity, while by (2) the force is also normal to the z-axis. Alternatively, by using (2) and (6) we have:

$$\vec{F}\cdot\vec{v}' = qB(v_y\hat{u}_x - v_x\hat{u}_y)\cdot(v_x\hat{u}_x + v_y\hat{u}_y) = qB(v_y v_x - v_x v_y) = 0\,.$$

It follows that the motion parallel to the *xy*-plane is uniform curvilinear, with speed equal to

$$v' = \left(v^2 - v_z{}^2\right)^{1/2} = \left(v_0{}^2 - v_{0z}{}^2\right)^{1/2} \equiv \text{constant}$$

where we have used the facts that $v=v_0$ and $v_z=v_{0z}$. Furthermore,

$$\vec{v}\times\vec{B} = (\vec{v}\,' + v_z\hat{u}_z)\times(B\hat{u}_z) = \vec{v}'\times\vec{B}\ ,$$

so that, by (1),

$$F = q\,|\,\vec{v}'\times\vec{B}\,| = qv'B \equiv \text{constant} \qquad (7)$$

If ρ' is the radius of curvature of the projection of the trajectory onto the *xy*-plane, then, given that F is purely centripetal, we have:

$$F = m\frac{v'^2}{\rho'} \;\;\Rightarrow\;\; \rho' = m\frac{v'^2}{F} \equiv \text{constant} \qquad (8)$$

(since both v' and F are constant). This means that the projection of the motion onto the *xy*-plane is uniform circular. Overall, the motion of the charge is the resultant of a uniform rectilinear motion parallel to the magnetic field, and a uniform circular motion on a plane perpendicular to the field. The trajectory is a *helix* (*uniform helical motion*). By (7) and (8) we get the radius of the circular projection of the motion:

$$\rho' = \frac{mv'}{qB} \qquad (9)$$

Then, by writing $v'=\omega\rho'$, we find that the angular velocity ω is again given by (5); that is, $\omega=qB/m$.

6. The total power radiated by a slowly moving accelerating charge is given by *Larmor's formula* (see Sec. 10.12 of [2])

$$P = \frac{q^2a^2}{6\pi\varepsilon_0 c^3} \qquad (10)$$

where a is the magnitude of the acceleration. Assuming that the charged particle is moving circularly on a plane normal to the magnetic field, and taking Eq. (3) into account, we have: $a=F/m=qvB/m$, where v is the constant speed of the particle. We observe that, for given values of q, v and B, the smaller the mass m of the particle, the greater the radiated power P and hence the greater the power losses. That is, radiation losses become increasingly significant as the mass of the particle decreases. Thus, for example, protons radiate far less than electrons in a cyclical accelerator.

7. The equation of motion of the charged particle is

$$m\frac{d\vec{v}}{dt}=q(\vec{v}\times\vec{B})\,.$$

By expanding the left-hand side into components, by using Eq. (2) for the right-hand side, and by equating corresponding components on the two sides of the equation, we obtain the following system of differential equations:

$$\frac{dv_x}{dt}=\omega v_y\ ,\quad \frac{dv_y}{dt}=-\omega v_x\ ,\quad \frac{dv_z}{dt}=0 \tag{11}$$

where we have put $\omega=qB/m$. Notice that the expression for ω is the same as that found previously for the angular velocity of the circular projection of the motion on the *xy*-plane.

The system (11) may be integrated by employing the methods described in [3] (cf., in particular, Sec. 4.1 and 5.1). The solution of the system is

$$v_x = A\cos(\omega t-\alpha)\ ,\quad v_y = -A\sin(\omega t-\alpha)\ ,\quad v_z=\lambda \tag{12}$$

where the $A>0$, α, λ are arbitrary constants. We notice that the speed of the particle is constant, equal to $v=(A^2+\lambda^2)^{1/2}$; the motion is thus uniform. The constants A, α, λ can be expressed in terms of the components (v_{0x} , v_{0y} , v_{0z}) of the initial velocity. Setting t=0 in (12), we find:

$$A=(v_{0x}^{\ 2}+v_{0y}^{\ 2})^{1/2}\ ,\quad \lambda=v_{0z}\ ,\quad \alpha=\arctan(v_{0y}/v_{0x})\,.$$

Relations (12) are rewritten as a system of differential equations:

$$dx/dt = A\cos(\omega t-\alpha)\ ,\quad dy/dt=-A\sin(\omega t-\alpha)\ ,\quad dz/dt=\lambda$$

the solution of which system is (by ignoring arbitrary constants)

$$x=(A/\omega)\sin(\omega t-\alpha)\ ,\quad y=(A/\omega)\cos(\omega t-\alpha)\ ,\quad z=\lambda t \tag{13}$$

Equations (13) express the coordinates of the particle as functions of time.

Projected to the *xy*-plane, the motion of the particle is uniform circular of radius $r=A/\omega$ and with angular velocity $\omega=qB/m$. Define now the function $\theta(t)$ by

$$\theta(t)\equiv\alpha-\omega t+\pi/2 \quad\Leftrightarrow\quad \omega t-\alpha=\pi/2-\theta(t)\,.$$

Equations (13) are then rewritten as

$$x=r\cos\theta(t)\ ,\quad y=r\sin\theta(t)\,.$$

We observe that the pair (r, θ) represents polar coordinates on the *xy*-plane, describing the circle $r=A/\omega=const.$ We also notice that, by its definition, $\theta(t)$ is a *decreasing* function of t ; that is, the polar angle θ decreases with time. This suggests that the circular projection of the path on the *xy*-plane is traversed in the *negative* direction, i.e., *clockwise*.

It also follows from (13) that the motion in the *z*-direction is uniform rectilinear. The overall path of the particle is a *helix* and the motion is, therefore, *uniform helical*.

References

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