Plane-wave solutions of Maxwell's equations: An educational note

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Synopsis

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

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Abstract

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

1. Introduction

Plane electromagnetic (e/m) waves constitute a significant type of solution of the time-dependent Maxwell equations. A standard educational approach in courses and textbooks (at both the intermediate [1-4] and the advanced [5,6] level; see also [7,8]) is to examine the prototype case of a monochromatic plane wave in both a conducting and a non-conducting medium.

In this note a more general approach to the problem is described that makes minimal initial assumptions regarding the specific functional forms of the plane waves representing the electric and the magnetic field. The only assumption one does need to make from the outset is that both fields (electric and magnetic) are expressible in integral form as linear superpositions of monochromatic waves. In particular, it is not even necessary to *a priori* require that the plane waves representing the two fields travel in the same direction.

In Section 2 we review the case of a monochromatic plane e/m wave in empty space. A more general (non-monochromatic) treatment of the plane-wave propagation problem in empty space is then described in Sec. 3. In Sec. 4 this general approach is extended to plane-wave solutions in the case of a conducting medium; an interesting difference from the monochromatic case is noted.

2. The monochromatic-wave description for empty space

In empty space, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units)

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &&(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &&(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. By applying the identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \ ,$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \ ,$$

we obtain separate wave equations for $\vec{E}$ and $\vec{B}$:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \qquad (2)$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \qquad (3)$$

where

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad (4)$$

We try monochromatic plane-wave solutions of (2) and (3), of angular frequency ω, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned} &\vec{E}(\vec{r},t)=\vec{E}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (a)\\ &\vec{B}(\vec{r},t)=\vec{B}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (b)\end{aligned} \qquad (5)$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes, and where

$$\frac{\omega}{k}=c \qquad (k=|\vec{k}|) \qquad (6)$$

The general solutions (5) do not *a priori* represent an e/m field. To find the extra constraints required, we must substitute Eqs. (5) into the Maxwell system (1). By taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, the *div* equations (1*a*) and (1*b*) yield

$$\vec{k}\cdot\vec{E}=0 \quad (a) \qquad \vec{k}\cdot\vec{B}=0 \quad (b) \qquad (7)$$

while the *rot* equations (1*c*) and (1*d*) give

$$\vec{k}\times\vec{E}=\omega\vec{B} \quad (a) \qquad \vec{k}\times\vec{B}=-\frac{\omega}{c^2}\vec{E} \quad (b) \qquad (8)$$

Now, we notice that the four equations (7)–(8) do not form an independent set since (7*b*) and (8*b*) can be reproduced by using (7*a*) and (8*a*). Indeed, taking the dot product of (8*a*) with $\vec{k}$ we get (7*b*), while taking the cross product of (8*a*) with $\vec{k}$, and using (7*a*) and (6), we find (8*b*).

So, from 4 independent Maxwell equations we obtained only 2 independent pieces of information. This happened because we “fed” our trial solutions (5) with more information than necessary, in anticipation of results that follow *a posteriori* from Maxwell’s equations. Thus, we assumed from the outset that the two waves (electric and magnetic) have similar simple functional forms and propagate in the same direction. By relaxing these initial assumptions, our analysis acquires a richer and much more interesting structure.

3. A more general approach for empty space

Let us assume, more generally, that the fields $\vec{E}$ and $\vec{B}$ represent plane waves propagating in empty space in the directions of the unit vectors $\hat{\tau}$ and $\hat{\sigma}$, respectively:

$$\vec{E}(\vec{r},t)=\vec{F}(\hat{\tau}\cdot\vec{r}-ct)\ ,\quad \vec{B}(\vec{r},t)=\vec{G}(\hat{\sigma}\cdot\vec{r}-ct) \tag{9}$$

Furthermore, assume that the functions $\vec{F}$ and $\vec{G}$ can be expressed as linear combinations of monochromatic plane waves of the form (5), for continuously varying values of k and ω, where $\omega=ck$, according to (6). Then $\vec{E}$ and $\vec{B}$ can be written in Fourier-integral form, as follows:

$$\begin{aligned}\vec{E}&=\int \vec{E}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}&=\int \vec{B}_0(k)\,e^{ik(\hat{\sigma}\cdot\vec{r}-ct)}dk\end{aligned} \tag{10}$$

In general, the integration variable k is assumed to run from 0 to $+\infty$. For notational economy, the limits of integration with respect to k will not be displayed explicitly.

By setting

$$u=\hat{\tau}\cdot\vec{r}-ct\ ,\quad v=\hat{\sigma}\cdot\vec{r}-ct \tag{11}$$

we write

$$\begin{aligned}\vec{E}(u)&=\int \vec{E}_0(k)\,e^{iku}dk\\ \vec{B}(v)&=\int \vec{B}_0(k)\,e^{ikv}dk\end{aligned} \tag{12}$$

We note that

$$\vec{\nabla}e^{iku}=ik\hat{\tau}e^{iku}\ ,\quad \vec{\nabla}e^{ikv}=ik\hat{\sigma}e^{ikv} \tag{13}$$

By using (12) and (13) we find that

$$\vec{\nabla}\cdot\vec{E}=\int ik\hat{\tau}\cdot\vec{E}_0(k)\,e^{iku}dk\ ,\quad \vec{\nabla}\cdot\vec{B}=\int ik\hat{\sigma}\cdot\vec{B}_0(k)\,e^{ikv}dk\ ,$$

$$\vec{\nabla}\times\vec{E}=\int ik\hat{\tau}\times\vec{E}_0(k)\,e^{iku}dk\ ,\quad \vec{\nabla}\times\vec{B}=\int ik\hat{\sigma}\times\vec{B}_0(k)\,e^{ikv}dk\ .$$

Moreover, we have that

$$\frac{\partial\vec{E}}{\partial t}=-\int i\omega\vec{E}_0(k)\,e^{iku}dk\ ,\quad \frac{\partial\vec{B}}{\partial t}=-\int i\omega\vec{B}_0(k)\,e^{ikv}dk$$

where, as always, $\omega=ck$.

The two Gauss' laws (1*a*) and (1*b*) yield

$$\int k\hat{\tau}\cdot\vec{E}_0(k)\,e^{iku}dk=0 \quad \text{and} \quad \int k\hat{\sigma}\cdot\vec{B}_0(k)\,e^{ikv}dk=0\,,$$

respectively. In order that these relations be valid identically for all *u* and all *v*, respectively, we must have

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\sigma}\cdot\vec{B}_0(k)=0\,, \;\text{ for all } k \tag{14}$$

From Faraday's law (1*c*) and the Ampère-Maxwell law (1*d*) we obtain two more integral equations:

$$\int k\hat{\tau}\times\vec{E}_0(k)\,e^{iku}dk=\int \omega\vec{B}_0(k)\,e^{ikv}dk \tag{15}$$

$$\int k\hat{\sigma}\times\vec{B}_0(k)\,e^{ikv}dk=-\int \frac{\omega}{c^2}\vec{E}_0(k)\,e^{iku}dk \tag{16}$$

where we have taken into account Eq. (4).

Taking the cross product of (15) with $\hat{\sigma}$ and using (16), we find the integral relation

$$\int k\,[(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0]\,e^{iku}dk=-\int k\vec{E}_0\,e^{iku}dk\,.$$

This is true for all *u* if

$$(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0=-\vec{E}_0 \;\Rightarrow\; (\hat{\sigma}\cdot\hat{\tau}-1)\vec{E}_0=(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}\,.$$

Given that, by (14), $\vec{E}_0$ and $\hat{\tau}$ are mutually perpendicular, the above relation can only be valid if $\hat{\sigma}\cdot\hat{\tau}=1$ and $\hat{\sigma}\cdot\vec{E}_0=0$. This, in turn, can only be satisfied if $\hat{\sigma}=\hat{\tau}$. The same conclusion is reached by taking the cross product of (16) with $\hat{\tau}$ and by using (15) as well as the fact that $\vec{B}_0$ is normal to $\hat{\sigma}$. From (11) we then have that

$$u=v=\hat{\tau}\cdot\vec{r}-ct$$

so that relations (12) become

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{iku}dk=\int\vec{E}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{iku}dk=\int\vec{B}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\end{aligned} \tag{17}$$

Equations (14) are now rewritten as

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0\,, \;\text{ for all } k \tag{18}$$

Furthermore, in order that (15) and (16) (with *u* and $\hat{\tau}$ in place of *v* and $\hat{\sigma}$, respectively) be identically valid for all *u*, we must have

$$k\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k)\;\Leftrightarrow\;\hat{\tau}\times\vec{E}_0(k)=c\vec{B}_0(k) \tag{19}$$

and

$$k\,\hat{\tau}\times\vec{B}_0(k)=-\frac{\omega}{c^2}\vec{E}_0(k)\;\Leftrightarrow\;\hat{\tau}\times\vec{B}_0(k)=-\frac{1}{c}\vec{E}_0(k) \tag{20}$$

for all k, where $k=\omega/c$. Notice, however, that (19) and (20) are not independent equations, since (20) is essentially the cross product of (19) with $\hat{\tau}$.

In summary, the general plane-wave solutions to the Maxwell system (1) are given by relations (17) with the additional constraints (18) and (19). This is, of course, a well-known result, derived here by starting with more general assumptions and by best exploiting the independence [9] of the Maxwell equations.

Let us summarize our main findings:

1. The fields $\vec{E}$ and $\vec{B}$ are plane waves traveling in the same direction, defined by the unit vector $\hat{\tau}$; these fields satisfy the Maxwell equations in empty space.

2. The e/m wave $(\vec{E},\vec{B})$ is a *transverse* wave. Indeed, from equations (17) and the orthogonality relations (18) it follows that

$$\hat{\tau}\cdot\vec{E}=0 \quad\text{and}\quad \hat{\tau}\cdot\vec{B}=0 \tag{21}$$

3. The fields $\vec{E}$ and $\vec{B}$ are mutually perpendicular. Moreover, the $(\vec{E},\vec{B},\hat{\tau})$ define a right-handed rectangular system. Indeed, by cross-multiplying (17) with $\hat{\tau}$ and by using (19) and (20), we find:

$$\hat{\tau}\times\vec{E}=c\vec{B}\;,\quad \hat{\tau}\times\vec{B}=-\frac{1}{c}\vec{E} \tag{22}$$

4. Taking *real values* of (21) and (22), we have:

$$\hat{\tau}\cdot\mathrm{Re}\vec{E}=0\;,\quad \hat{\tau}\cdot\mathrm{Re}\vec{B}=0 \quad\text{and}\quad \hat{\tau}\times\mathrm{Re}\vec{E}=c\,\mathrm{Re}\vec{B} \tag{23}$$

The magnitude of the last vector equation in (23) gives a relation between the instantaneous values of the electric and the magnetic field:

$$|\mathrm{Re}\vec{E}|=c\,|\mathrm{Re}\vec{B}| \tag{24}$$

The above results for empty space can be extended in a straightforward way to the case of a *linear, non-conducting, non-dispersive* medium upon replacement of ε_0 and μ_0 with ε and μ, respectively [3]. The (frequency-independent) speed of propagation of the plane e/m wave in this case is $\upsilon=1/(\varepsilon\mu)^{1/2}$.

4. The case of a conducting medium

The Maxwell equations for a conducting medium of conductivity σ may be written as follows [1,3]:

$$(a) \quad \vec{\nabla}\cdot\vec{E}=0 \qquad (c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$(b) \quad \vec{\nabla}\cdot\vec{B}=0 \qquad (d) \quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\frac{\partial\vec{E}}{\partial t} \tag{25}$$

By using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}\ ,$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}\,,$$

the relations (25) lead to the *modified wave equations*

$$\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \tag{26}$$

$$\nabla^2\vec{B}-\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\frac{\partial\vec{B}}{\partial t}=0 \tag{27}$$

Guided by our monochromatic-wave approach to the problem in [7,8], we now try a more general, integral form of solution of the above wave equations:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\,\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{E}_0(k)\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\,\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{B}_0(k)\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk\end{aligned} \tag{28}$$

where s is a real parameter related to the conductivity of the medium. As in the vacuum case, the unit vector $\hat{\tau}$ indicates the direction of propagation of the wave. Notice that we have assumed from the outset that both waves – electric and magnetic – propagate in the same direction, in view of the fact that our results must agree with those for a non-conducting medium (in particular, for the vacuum) upon setting s=0.

It is convenient to set

$$\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\equiv A(\vec{r},t) \tag{29}$$

Then, Eq. (28) takes on the form

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,A(\vec{r},t)\,dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,A(\vec{r},t)\,dk\end{aligned} \tag{30}$$

The following relations can be easily proven:

$$\vec{\nabla}A(\vec{r},t)=(ik-s)\,\hat{\tau}A(\vec{r},t) \tag{31}$$

$$\nabla^2A(\vec{r},t)=(s^2-k^2-2isk)A(\vec{r},t) \tag{32}$$

Moreover,

$$\frac{\partial}{\partial t}A(\vec{r},t)=-i\omega A(\vec{r},t) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}A(\vec{r},t)=-\omega^2 A(\vec{r},t)\ .$$

From (26) we get

$$\int [(s^2-k^2+\varepsilon\mu\omega^2)+i(\mu\sigma\omega-2sk)]\,\vec{E}_0(k)A(\vec{r},t)\,dk=0$$

[a similar integral relation is found from (27)]. This will be identically satisfied for all $\vec{r}$ and t if

$$s^2-k^2+\varepsilon\mu\omega^2=0 \quad \text{and} \quad \mu\sigma\omega-2sk=0 \tag{33}$$

By using relations (33), ω and s can be expressed as functions of k, as required in order that the integral relations (28) make sense. Notice, in particular, that, by the second relation (33), s=0 if σ=0 (non-conducting medium). Then, by the first relation, $\omega/k=1/(\varepsilon\mu)^{1/2}$, which is the familiar expression for the speed of propagation of an e/m wave in a non-conducting medium [3].

From the two Gauss' laws (25*a*) and (25*b*) we get the corresponding integral relations

$$\int (ik-s)\,\hat{\tau}\cdot\vec{E}_0(k)A(\vec{r},t)\,dk=0\ ,$$
$$\int (ik-s)\,\hat{\tau}\cdot\vec{B}_0(k)A(\vec{r},t)\,dk=0\ .$$

These will be identically satisfied for all $\vec{r}$ and t if

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0,\ \text{for all } k \tag{34}$$

From (25*c*) and (25*d*) we find

$$\int (ik-s)\,\hat{\tau}\times\vec{E}_0(k)A(\vec{r},t)\,dk=\int i\omega\vec{B}_0(k)A(\vec{r},t)\,dk$$

and

$$\int (ik-s)\,\hat{\tau}\times\vec{B}_0(k)A(\vec{r},t)\,dk=\int (\mu\sigma-i\varepsilon\mu\omega)\vec{E}_0(k)A(\vec{r},t)\,dk\ ,$$

respectively. To satisfy these for all $\vec{r}$ and t, we require that

$$(k+is)\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k) \tag{35}$$

and

$$(k+is)\,\hat{\tau}\times\vec{B}_0(k)=-(\varepsilon\mu\omega+i\mu\sigma)\vec{E}_0(k) \tag{36}$$

Note, however, that (36) is not an independent equation since it can be reproduced by cross-multiplying (35) with $\hat{\tau}$ and by taking into account Eqs. (33) and (34).

We note the following:

1. From (30) and (34) we have that

$$\hat{\tau}\cdot\vec{E}=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}=0 \tag{37}$$

or, in real form, $\hat{\tau}\cdot\mathrm{Re}\,\vec{E}=0$ and $\hat{\tau}\cdot\mathrm{Re}\,\vec{B}=0$. This means that both $\mathrm{Re}\,\vec{E}$ and $\mathrm{Re}\,\vec{B}$ are normal to the direction of propagation of the wave.

2. From (30) and (35) we get

$$\hat{\tau}\times\vec{E}=\int\frac{\omega}{k+is}\vec{B}_0(k)A(\vec{r},t)\,dk \tag{38}$$

The integral on the right-hand side of (38) is, generally, not a vector parallel to $\vec{B}$. Now, in the limit of negligible conductivity (σ=0) the relations (33) give s=0 and $\omega/k=1/(\varepsilon\mu)^{1/2}$. The ratio ω/k represents the speed of propagation υ in the non-conducting medium, for the frequency ω. If the medium is *non-dispersive*, the speed $\upsilon=\omega/k$ is constant, independent of frequency. Then Eq. (38) (with s=0) becomes

$$\hat{\tau}\times\vec{E}=\upsilon\int\vec{B}_0(k)A(\vec{r},t)\,dk=\upsilon\vec{B}$$

and, in real form, it reads $\hat{\tau}\times\mathrm{Re}\vec{E}=\upsilon\,\mathrm{Re}\vec{B}$. Geometrically, this means that the $(\mathrm{Re}\,\vec{E},\mathrm{Re}\,\vec{B},\hat{\tau})$ define a right-handed rectangular system.

3. As shown in [7,8], the $\vec{E}$ and $\vec{B}$ are always mutually perpendicular in a *monochromatic* e/m wave of definite frequency ω, traveling in a conducting medium. Such a wave is represented in real form by the equations

$$\vec{E}(\vec{r},t)=\vec{E}_0\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\alpha)\ ,$$
$$\vec{B}(\vec{r},t)=\frac{\sqrt{k^2+s^2}}{\omega}(\hat{\tau}\times\vec{E}_0)\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\beta)$$

where $\vec{E}_0$ is a real vector and where $\tan(\beta-\alpha)=s/k$. This perpendicularity between $\vec{E}$ and $\vec{B}$ ceases to exist, however, in a non-monochromatic wave of the form (28).

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