Second-order linear differential equations and application to oscillations

C. J. Papachristou

Department of Physical Sciences, Hellenic Naval Academy

papachristou@hna.gr

1. Second-order linear differential equations

A second-order linear differential equation (DE) has the general form

$$y'' + a(x)\,y' + b(x)\,y = f(x) \tag{1}$$

where $y=y(x)$ and where $a(x)$, $b(x)$, $f(x)$ are given functions. If $f(x)\equiv 0$, the DE (1) is called *homogeneous linear* :

$$y'' + a(x)\,y' + b(x)\,y = 0 \tag{2}$$

As is easy to prove, if a function $y_1(x)$ is a solution of (2), then so is the function $y_2(x)=Cy_1(x)$ (C=const.). More generally, the following is true:

Theorem 1: If $y_1(x)$, $y_2(x)$,… are solutions of the homogeneous DE (2), then every linear combination of the form $y=C_1\,y_1(x)+C_2\,y_2(x)+\dots$ (where C_1 , C_2 ,… are constants) also is a solution of (2).

Proof: By substituting for y on the left-hand side of (2) and by taking into account that each of the $y_1(x)$, $y_2(x)$,… satisfies this DE, we have:

$$y'' + a(x)\,y' + b(x)\,y = C_1\,(y_1'' + a\,y_1' + b\,y_1) + C_2\,(y_2'' + a\,y_2' + b\,y_2) + \dots = 0 \,.$$

Let $y_1(x)$ and $y_2(x)$ be two non-vanishing solutions of the homogeneous DE (2) [notice that the zero function $y(x)\equiv 0$ is a particular solution of (2)]. We say that the functions y_1 and y_2 are *linearly independent* if one is not a scalar multiple of the other. To put it in more formal terms, linear independence of y_1 and y_2 means that a relation of the form $C_1\,y_1(x)+C_2\,y_2(x)\equiv 0$ can only be true if $C_1=C_2=0$.

If we manage to find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous DE (2) (I can assure you that no other solution linearly independent of the former two exists!) then the *general solution* of (2) is the linear combination

$$y = C_1\,y_1(x) + C_2\,y_2(x) \tag{3}$$

where C_1 , C_2 are arbitrary constants.

Theorem 2: The general solution of the non-homogeneous DE (1) is the sum of the general solution (3) of the corresponding homogeneous equation (2) and *any particular solution* of (1).

Analytically: Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the homogeneous DE (2), and let $y_0(x)$ be any particular solution of (1). Then, the general solution of (1) is

$$y = C_1 y_1(x) + C_2 y_2(x) + y_0(x) \tag{4}$$

This practically means that, for any chosen y_0, any other particular solution of (1) can be derived from (4) by properly choosing the constants C_1 and C_2. Since (4) contains the totality of particular solutions of (1), it must be the general solution of (1).

2. Homogeneous linear equation with constant coefficients

This DE has the form

$$y'' + a y' + b y = 0 \tag{5}$$

with constant a and b. It will be assumed that a and b are real numbers.

Theorem 3: If the complex function $y=u(x)+iv(x)$ satisfies the DE (5), then the same is true for each of the real functions $y_1=u(x)$ and $y_2=v(x)$ (real and imaginary part of y, respectively).

Proof: Putting $y=u+iv$ into (5), we find:

$$(u'' + a u' + b u) + i (v'' + a v' + b v) = 0 ,$$

which is true iff $u''+a u'+b u = 0$ and $v''+a v'+b v = 0$.

The standard method for solving (5) is the following: We try an exponential solution of the form $y=e^{kx}$. Then, $y'=ke^{kx}$, $y''=k^2e^{kx}$, and (5) yields (after eliminating e^{kx}):

$$k^2 + ak + b = 0 \quad (\textit{characteristic equation}) \tag{6}$$

We distinguish the following cases:

1. Eq. (6) has real and distinct roots k_1, k_2. Then, the functions $e^{k_1 x}$ and $e^{k_2 x}$ are linearly independent and, according to (3), the general solution of (5) is of the form

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} \tag{7}$$

2. Eq. (6) has real and equal roots, $k_1 = k_2 \equiv k$. The general solution of (5) is, in this case (check!),

$$y = (C_1 + C_2 x) e^{kx} \tag{8}$$

3. Eq. (6) has complex conjugate roots $k_1=\alpha+i\beta$, $k_2=\alpha-i\beta$ (where α, β are real). The general solution of (5) is

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} = e^{\alpha x}(C_1 e^{i\beta x} + C_2 e^{-i\beta x}) .$$

By Euler's formula, $e^{\pm i\beta x} = \cos\beta x \pm i \sin\beta x$. We thus have:

$$y = e^{\alpha x}[(C_1 + C_2)\cos\beta x + i\,(C_1 - C_2)\sin\beta x] .$$

Since the (generally complex) constants C_1 and C_2 are arbitrary, we may put C_1 in place of C_1+C_2 and C_2 in place of $i\,(C_1 - C_2)$, so that, finally,

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) \qquad (9)$$

In any case, the general solution of (5) contains two arbitrary constants C_1 and C_2 . Upon assigning specific values to C_1 and C_2 we get a *particular solution* of (5). The values of C_1 and C_2 (and thus the particular solution itself) are determined from the general solution if we are given two *initial conditions* that the sought-for particular solution must obey. There are two kinds of initial conditions:

(*a*) We are given the values of $y(x)$ and $y'(x)$ for some value $x=x_0$ of x.

(*b*) We are given the values of $y(x)$ for $x=x_1$ and $x=x_2$.

Examples:

1. $y'' - y' - 2y = 0 \Rightarrow a = -1,\ b = -2$. The characteristic equation (6) is written: $k^2 - k - 2 = 0$, with real roots $k_1=2$, $k_2=-1$. The general solution (7) is $y = C_1 e^{2x} + C_2 e^{-x}$. Assume the initial conditions $y=2$ and $y'=-5$ when $x=0$. Then, $C_1=-1$, $C_2=3$ (show it!) and we get the *particular* solution $y = -e^{2x} + 3e^{-x}$.

2. $y'' - 6y' + 9y = 0 \Rightarrow a = -6,\ b = 9$. The characteristic equation (6) is written: $k^2 - 6k + 9 = 0$, with real and equal roots $k_1=k_2=3$. The general solution (8) is $y = (C_1 + C_2 x)\, e^{3x}$.

3. $y'' - 4y' + 13y = 0 \Rightarrow a = -4,\ b = 13$. The characteristic equation (6) is written: $k^2 - 4k + 13 = 0$, with complex conjugate roots $k_1=2+3i$, $k_2=2-3i$. The general solution (9) is (with $\alpha=2$, $\beta=3$): $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$. (Show that essentially the same result is found by making the alternative choice $\alpha=2$, $\beta=-3$.)

3. Harmonic oscillation

In a harmonic oscillation along the x-axis the total force on the oscillating body (of mass m) is $F=-kx$ ($k>0$), where x is the momentary displacement of the body from the position of equilibrium ($x=0$). By Newton's second law we have that $F=ma$, where a is the acceleration of the body: $a=d^2x/dt^2$. Therefore,

$$m\, d^2x/dt^2 = -kx$$

or, setting $k/m \equiv \omega^2$ (where we assume that $\omega>0$),

$$x'' + \omega^2 x = 0 \qquad (10)$$

Eq. (10) is a homogeneous linear DE of the form (5) with x in place of y and t in place of x (notice that the first-derivative term is missing in this case). The characteristic equation (6) is written: $k^2+\omega^2=0$ (or, analytically, $k^2+0k+\omega^2=0$), with complex roots $k=\pm i\omega$ (analytically, $k_1=0+i\omega$, $k_2=0-i\omega$). The general solution of (10) is given by (9), with $\alpha=0$ and $\beta=\omega$:

$$x = C_1 \cos \omega t + C_2 \sin \omega t \qquad (11)$$

where we assume that the constant coefficients C_1 and C_2 are real in order for the solution (11) to have physical meaning.

The general solution (11) can be put in different but equivalent form by setting

$$C_1 = A \sin\varphi\ ,\ \ C_2 = A\cos\varphi\ \ (A>0) \ \ \Leftrightarrow\ \ A=(C_1^{\,2}+C_2^{\,2})^{1/2}\ ,\ \ \tan\varphi = C_1/C_2\ .$$

Then,

$$x = A \sin(\omega t + \varphi) \qquad (12)$$

The positive constant A is called the *amplitude* of the oscillation, while the angle φ is called the *initial phase* (the value of the *phase* $\omega t+\varphi$ at time $t=0$). The positive constant ω is the *angular frequency* of oscillation, to be called just "*frequency*" in the sequel.

Notice that, if we set $C_1=A\cos\varphi$, $C_2=-A\sin\varphi$ in (11), we will get the general solution of (10) in the form

$$x = A\cos(\omega t + \varphi) \qquad (13)$$

which is equivalent to (12). Indeed, equation (13) follows directly from (12) by putting $\varphi+(\pi/2)$ in place of φ (which is arbitrary anyway) in the latter equation.

4. Damped oscillation

In a damped oscillation, in addition to the restoring force $-kx$, opposite to the displacement x from the equilibrium position, there is a frictional force $-\lambda v= -\lambda dx/dt$ ($\lambda>0$) opposite to the velocity v. The total force on the body is $F= -kx-\lambda dx/dt$. By Newton's law, $F=m\,d^2x/dt^2$. Hence,

$$m\,d^2x/dt^2 = -kx - \lambda\,dx/dt\,.$$

We set

$$k/m \equiv \omega_0^2 \;\; (\omega_0 = \textit{natural frequency of oscillation} \text{ without damping}), \quad \lambda/m \equiv 2\gamma,$$

so that

$$x'' + 2\gamma x' + \omega_0^2 x = 0 \tag{14}$$

Eq. (14) is a homogeneous linear DE. The characteristic equation (6) is

$$k^2 + 2\gamma k + \omega_0^2 = 0 \quad \Rightarrow \quad k = -\gamma \pm (\gamma^2 - \omega_0^2)^{1/2}\,.$$

We distinguish the following cases:

1. *Large damping* $\Leftrightarrow$ $\gamma > \omega_0$. We have two real solutions:

$$k_1 = -\gamma + (\gamma^2 - \omega_0^2)^{1/2}\,, \quad k_2 = -\gamma - (\gamma^2 - \omega_0^2)^{1/2}\,.$$

The general solution of (14) is of the form (7):

$$x = C_1\,e^{k_1 t} + C_2\,e^{k_2 t} \tag{15}$$

Let us assume that $C_1>0$ and $C_2>0$. Given that $k_1<0$ and $k_2<0$ (why?) we see that $x>0$ at all times t και, moreover, $x\to 0$ as $t\to\infty$. That is, as the time t increases, the moving object approaches the equilibrium position $x=0$ without ever crossing it. The motion is therefore *non-oscillatory*.

2. *Critical damping* $\Leftrightarrow$ $\gamma = \omega_0$. Then, $k_1 = k_2 = -\gamma$, and the general solution of (14) is of the form (8):

$$x = (C_1 + C_2\,t)\,e^{kt} = (C_1 + C_2\,t)\,e^{-\gamma t} \tag{16}$$

If we assume that $C_1>0$ and $C_2>0$, we see again that $x>0$ at all t and that $x\to 0$ as $t\to\infty$. (For the term $t\,e^{-\gamma t} = t\,/\,e^{\gamma t}$ we may use L'Hospital's rule for the indeterminate form ∞/∞; show this!) Thus, there is no oscillation in this case either.

3. *Small damping* $\Leftrightarrow$ $\gamma < \omega_0$. We have two complex conjugate solutions:

$$k = -\gamma \pm i\,\omega_1 \quad \text{where} \quad \omega_1 = (\omega_0^2 - \gamma^2)^{1/2}\,.$$

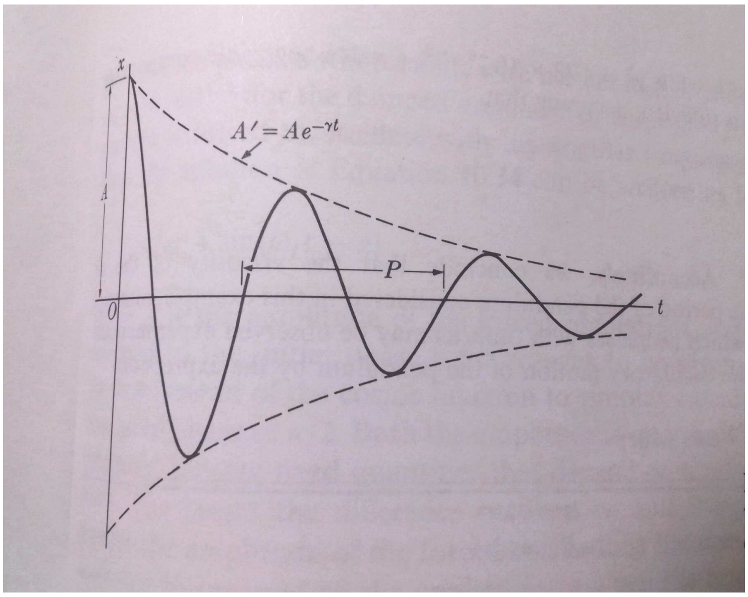
The general solution will be of the form (9), with $\alpha = -\gamma$ and $\beta = \omega_1$:

$$x = e^{-\gamma t} (C_1 \cos \omega_1 t + C_2 \sin \omega_1 t) ,$$

or, by setting $C_1 = A \sin \varphi$, $C_2 = A \cos \varphi$ (A>0),

$$x = A\, e^{-\gamma t} \sin (\omega_1 t + \varphi) \qquad (17)$$

We notice that the amplitude $Ae^{-\gamma t}$ decreases exponentially with time.



5. Forced oscillation

In a forced oscillation, in addition to the restoring force $-kx$ and the frictional force $-\lambda v = -\lambda dx/dt$ the body is subject to an external force of the form

$$F(t) = F_0 \sin \omega_f t \quad (F_0 > 0) .$$

The total force on the body is $F = -kx - \lambda dx/dt + F_0 \sin \omega_f t$. By Newton's law we have that

$$m\, d^2x / dt^2 = -kx - \lambda\, dx/dt + F_0 \sin \omega_f t .$$

We set

$$k/m \equiv \omega_0^2 \ (\omega_0 = \textit{natural frequency}), \quad \lambda/m \equiv 2\gamma, \quad F_0 / m \equiv f_0 ,$$

so that

$$x'' + 2\gamma x' + \omega_0^2 x = f_0 \sin \omega_f t \qquad (18)$$

Eq. (18) is a non-homogeneous linear DE. According to Theorem 2 of Sec. 1, its general solution is the sum of the general solution of the corresponding homogeneous equation,

$$x'' + 2\gamma x' + \omega_0^2 x = 0 \ ,$$

and *any particular solution* of (18). For small damping ($\gamma < \omega_0$) the general solution of the homogeneous equation is given by (17):

$$x = A_1 e^{-\gamma t} \sin(\omega_1 t + \varphi_1) \quad \text{where} \quad \omega_1 = (\omega_0^2 - \gamma^2)^{1/2} \, .$$

As can be verified, a particular solution of (18) is the following:

$$x = A \sin(\omega_f t + \varphi) \tag{19}$$

where

$$A = \frac{f_0}{\left[\left(\omega_f^{\,2} - \omega_0^{\,2}\right)^2 + 4\gamma^2 \omega_f^{\,2}\right]^{1/2}} \quad \text{and} \quad \tan\varphi = \frac{2\gamma\omega_f}{\omega_f^{\,2} - \omega_0^{\,2}} \tag{20}$$

The general solution of (18) is, therefore,

$$x = A_1 e^{-\gamma t} \sin(\omega_1 t + \varphi_1) + A \sin(\omega_f t + \varphi) \tag{21}$$

with *arbitrary* A_1, φ_1. The first term on the right in (21) decreases exponentially with time and dies out quickly. In a steady-state situation, therefore, what remains is the particular solution (19):

$$x = A \sin(\omega_f t + \varphi) \, .$$

The amplitude A of oscillation is a function of the applied frequency ω_f, according to (20). This amplitude attains a maximum value when the denominator in the first relation (20) becomes minimum. This occurs when

$$\omega_f = (\omega_0^2 - 2\gamma^2)^{1/2} \equiv \omega_A \tag{22}$$

Proof: We set $\omega_f \equiv \omega$, for simplicity, and we consider the function

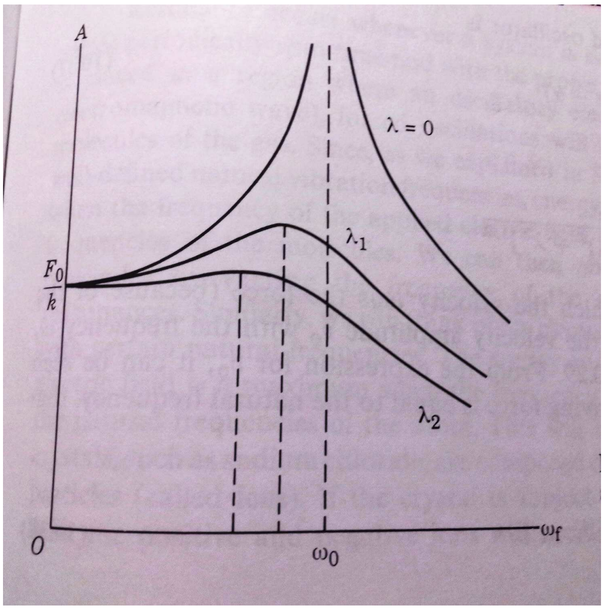
$$\Psi(\omega) = (\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2 \, ,$$

so that $A = f_0 / [\Psi(\omega)]^{1/2}$. We can show that

$$\Psi'(\omega) = 0 \ \text{ for } \ \omega = (\omega_0^2 - 2\gamma^2)^{1/2} = \omega_A \quad \text{and} \quad \Psi''(\omega_A) = 8\omega_A^2 > 0 \, .$$

Thus, for small damping ($2\gamma^2 < \omega_0^2$) the function $\Psi(\omega)$ is *minimum*, hence the amplitude A is *maximum*, when $\omega_f = \omega_A$. This situation is called *amplitude resonance*.

In the following figure it is assumed that $\lambda_1<\lambda_2 \Leftrightarrow \gamma_1<\gamma_2$. This means that, in accordance with (22), $\omega_{A,1} > \omega_{A,2}$. In the case of no damping ($\lambda=0 \Leftrightarrow \gamma=0$) Eq. (22) yields $\omega_A=\omega_0$. In other words, in an *undamped* forced oscillation the amplitude becomes maximum (in fact, infinite) when the applied frequency ω_f is equal to the natural frequency ω_0 of oscillation.



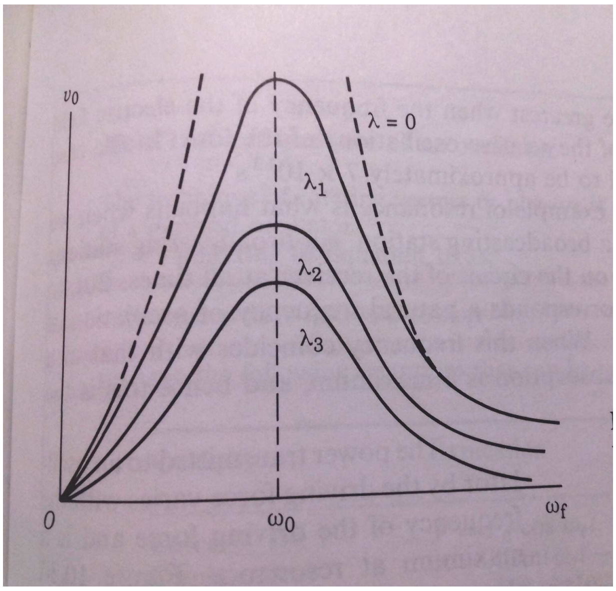
By differentiating (19) we find the velocity of the oscillating body:

$$v = dx/dt = \omega_f A \cos(\omega_f t + \varphi) \equiv v_0 \cos(\omega_f t + \varphi)$$

where, by (20),

$$v_0 = \omega_f A = \frac{f_0}{\left[\left(1-\dfrac{\omega_0^2}{\omega_f^2}\right)^2 + 4\gamma^2\right]^{1/2}} \ .$$

The velocity amplitude v_0 becomes maximum when the denominator on the right is minimum, which occurs for $\omega_f = \omega_0$. The kinetic energy $mv_0^2/2$ then reaches its maximum value and there is *energy resonance*.



Note that, in contrast to amplitude resonance, the frequency ω_f for energy resonance is independent of the damping factor λ and is always equal to the *natural frequency* ω_0 of the oscillator. At this frequency the work supplied by the external force $F(t)$ to the oscillator per unit time is maximum. That is, the oscillator absorbs the largest possible power from the external agent that exerts the force F.

Notice also that, in the case of zero damping ($\lambda=0 \Leftrightarrow \gamma=0$) the velocity amplitude v_0 becomes *infinite* at energy resonance, i.e., for $\omega_f=\omega_0$. This rather unphysical situation is, of course, purely theoretical since a mechanical motion with no friction whatsoever is practically impossible!