C. J. Papachristou

One-Dimensional Newtonian Systems

- **Conservative and Periodic Systems**
- **Oscillatory Systems**

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C. J. Papachristou

Department of Physical Sciences, Hellenic Naval Academy

papachristou@hna.gr

The cases of conservative and oscillatory Newtonian systems in one dimension are studied. Certain unique properties of simple harmonic motion are noted.

A. One-dimensional conservative systems

1. The general solution to the problem

Consider a particle of mass *m*, moving along the *x*-axis under the action of a total force $F(x)$. The position $x(t)$ of the particle as a function of time is found by integrating the second-order differential equation (Newton's second law)

$$m\, d^2x/dt^2 = F(x) \tag{1}$$

for given initial conditions $x(t_0)=x_0$, $v(t_0)=v_0$, where $v=dx/dt$ is the velocity of the particle.

Define the auxiliary function $U(x)$ (potential energy of the particle) by

$$U(x) = -\int_0^x F(x')dx' \iff F(x) = -dU/dx \tag{2}$$

Then (1) is written

$$m\, d^2x/dt^2 + dU/dx = 0 \,.$$

We multiply by $v=dx/dt$, which plays the role of an integrating factor:

$$(dx/dt)\,(m\, d^2x/dt^2 + dU/dx) = 0 \,.$$

By noticing that

$$(dx/dt)\,(m\, d^2x/dt^2) = v\,(m\, dv/dt) = (d/dt)\,(m v^2/2)$$

and that $(dx/dt)\,(dU/dx) = dU/dt$, we have: $(d/dt)\,(m v^2/2 + U) = 0 \;\Rightarrow$

$$m v^2/2 + U(x) \equiv T + U = E = const. \tag{3}$$

(where T = kinetic energy) which expresses conservation of total mechanical energy.

From relation (3) we get

$$(dx/dt)^2 = (2/m)\,[E-U(x)] \;\Rightarrow\; dx/dt = \pm\{(2/m)\,[E-U(x)]\}^{1/2} \,.$$

Integrating this first-order differential equation and taking into account the initial condition $x=x_0$ for $t=t_0$, we have:

$$\int_{x_0}^{x} \frac{\pm\, dx}{\left\{ \dfrac{2}{m} \left[E - U(x) \right] \right\}^{1/2}} = t - t_0 \tag{4}$$

where the plus sign is chosen for motion in the *positive* direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the *negative* direction ($v<0$, $x<x_0$).

The value of the constant E may be determined by applying the given initial conditions to (3):

$$E = m v_0^2/2 + U(x_0) \tag{5}$$

(although, as we will see, other physical considerations may also be used).

2. The case of periodic motion

Let us now assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x \neq 0$ (this arrangement is always possible because of the arbitrariness in the definition of the zero-level of the potential energy). In general, the graph of $U(x)$ need not be symmetric with respect to the axis $x=0$.

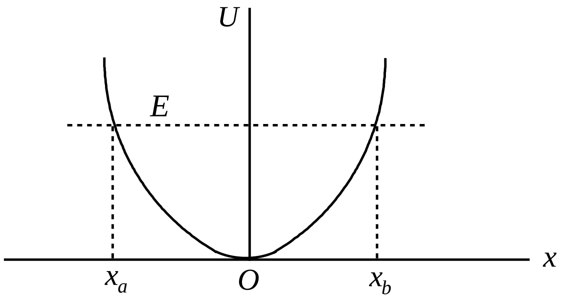

Fig. 1

Let E be the total mechanical energy of the particle. Since $E=T+U$ with $T \geq 0$, it follows that $E \geq U(x)$ for any physical motion. The motion is thus *bounded* between the points x_a and x_b of the x-axis, these points being *turning points* at which the particle stops momentarily ($E=U \Rightarrow T=0 \Rightarrow v_a=v_b=0$). The time it takes for a complete journey from x_a to x_b and back to x_a is found by using (4) with the appropriate sign for each direction of motion:

$$P = \int_{x_a}^{x_b} \frac{dx}{\{\cdots\}^{1/2}} + \int_{x_b}^{x_a} \frac{-dx}{\{\cdots\}^{1/2}} \;\Rightarrow$$

$$P = 2 \int_{x_a}^{x_b} \frac{dx}{\left\{ \dfrac{2}{m} \left[E - U(x) \right] \right\}^{1/2}} \tag{6}$$

Since P is fixed for given x_a and x_b , the motion is *periodic* with *period* P. Generally, the period depends on the limits of integration x_a and x_b and therefore it depends on the total energy E of the particle. An exception where P does *not* depend on E is *simple harmonic motion*, as we now show.

3. Simple harmonic motion (SHM)

In SHM the potential energy is of *parabolic* form: $U(x)=kx^2/2$, which is symmetric with respect to the axis $x=0$ (see Fig. 1). The total force is a *restoring force* given by

$$F(x) = -\, dU/dx = -\, kx \tag{7}$$

If frictional (damping) forces are present, the total force also contains a velocity-dependent term $-\lambda v= -\lambda dx/dt$ and the system is no longer conservative.

According to Fig. 1 the motion takes place between $x_a= -A$ and $x_b=A$, where $A\geq 0$ is the *amplitude* of oscillation. At the two extreme points the kinetic energy T vanishes momentarily and the total energy, which is equal to $E=T+U$ and which retains a fixed value during the motion, is equal to the potential energy: $E=U(\pm A)=kA^2/2$. Since E is the same at all points x, we conclude that

$$E = m v^2/2 + k x^2/2 = k A^2/2 \tag{8}$$

The period of oscillation is found by using (6):

$$P = 2\int_{-A}^{A} \left\{ \frac{2}{m}(E - kx^2/2) \right\}^{-1/2} dx \,.$$

Substituting for E from (8), we find:

$$P = \frac{2}{\omega}\int_{-A}^{A} \left(A^2 - x^2 \right)^{-1/2} dx$$

where we have set $\omega=(k/m)^{1/2}$ (angular frequency). Putting $x/A=u$ and using the integral formula

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

we finally find (see Appendix):

$$P = 2\pi/\omega = 2\pi\,(m/k)^{1/2} \,.$$

We conclude that, if the potential energy is of parabolic form: $U(x)=kx^2/2$, the period P of motion is independent of the amplitude A, thus independent of the total energy $E=kA^2/2$.

But, what if $U(x)$ is like that in Fig. 1 but *not* parabolic? For example, let U be of the form $U(x)=\lambda x^4/4$, so that $F(x)= -dU/dx = -\lambda x^3$. Since $U(x)$ is symmetric with respect to the axis $x=0$, the periodic motion will take place between the points $x_a= -A$ and $x_b=A$ and the total energy will be equal to $E=U(\pm A)=\lambda A^4/4$. The period is

$$P = 2\int_{-A}^{A} \left\{ \frac{2}{m}(E - \lambda x^4/4) \right\}^{-1/2} dx = \frac{2}{\mu}\int_{-A}^{A} \left(A^4 - x^4 \right)^{-1/2} dx = \frac{2}{\mu A}\int_{-1}^{1} \frac{du}{\sqrt{1-u^4}}$$

where we have set $u=x/A$ and $\mu=(\lambda/2m)^{1/2}$. Obviously, P depends on the amplitude A, thus on the total energy E. (A more general proof regarding non-parabolic potential energies, in general, is given in the Appendix.)

Returning to SHM, we may obtain the equation of motion $x=x(t)$ by using (4) with $U(x)=kx^2/2$ and $E=kA^2/2$. Let us assume first that the motion is in the positive direction, so that $x>x_0$. Setting $\omega=(k/m)^{1/2}$, we have:

$$\int_{x_0}^{x} \left(A^2 - x^2\right)^{-1/2} dx = \omega\,(t-t_0)\,.$$

Using the integral formula

$$\int \left(A^2 - x^2\right)^{-1/2} dx = \arcsin(x/A) + C$$

and making appropriate substitutions for constants, we find an equation of the form[1]

$$\arcsin(x/A) = \omega t+\alpha \quad \Rightarrow \quad x = A\sin(\omega t+\alpha)\,.$$

For motion in the negative direction ($x<x_0$) we choose the minus sign in (4), so that

$$\int_{x_0}^{x} \left(A^2 - x^2\right)^{-1/2} dx = -\,\omega\,(t-t_0)\,.$$

This yields a result of the form[2]

$$\arcsin(x/A) = -\,\omega t+\beta \quad \Rightarrow \quad x = -A\sin(\omega t-\beta)\,.$$

Since the constant β is arbitrary (being dependent on the arbitrary constants x_0 and t_0) we may set $-\beta \equiv \pi+\alpha$, so that $x = A\sin(\omega t+\alpha)$, as before.

Thus, the general solution for SHM is $x(t) = A\sin(\omega t+\alpha)$. Physically, A is the *amplitude* of oscillation, ω is the *angular frequency* and α is the *initial phase* (i.e., the *phase* $\omega t+\alpha$ at $t=0$).

4. Motion under a constant force of gravity

A projectile of mass m is fired straight upward at time $t_0=0$ from the point $x=0$ of the vertical x-axis, with initial velocity $v_0>0$ (we choose the positive direction of the x-axis to be upward). The constant acceleration of gravity is directed downward, so that $a=dv/dt=-g$. The total force on the particle (assuming no air resistance) and the corresponding potential energy of the particle are given by

$$F(x) = ma = -mg \quad \Leftrightarrow \quad U(x) = mgx \quad \text{[we assume that } U(0)=0\text{]}\,.$$

Relation (4) (with the plus sign for upward motion) is written

$$\int_0^x \frac{dx}{(E-mgx)^{1/2}} = (2/m)^{1/2}\,t\ .$$

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[1] Explicitly: $\alpha = \arcsin(x_0/A) - \omega t_0$.

[2] Explicitly: $\beta = \arcsin(x_0/A) + \omega t_0$.

By (5) and by using the initial conditions we have that $E=mv_0^2/2+U(0)=mv_0^2/2$ (since $U=0$ for $x_0=0$). Thus, the requirement $E-mgx \geq 0$ yields $x \leq v_0^2/2g$. Physically this means that the particle will reach a maximum height $h=v_0^2/2g$ where it will stop momentarily before it starts to move downward (i.e., in the negative direction).

With this restriction on the acceptable values of x, the integration may be performed to give

$$(E-mgx)^{1/2} = E^{1/2} - (m/2)^{1/2}\, g\, t\ .$$

Squaring this, we find:

$$x = (2E/m)^{1/2}\, t - g\, t^2/2\ .$$

But, $E = m\, v_0^2/2 \ \Rightarrow (2E/m)^{1/2} = v_0$ (since $v_0 > 0$). Thus, finally,

$$x = v_0\, t - g\, t^2/2$$

which is, of course, a familiar result.

5. Phase curves of a one-dimensional conservative system

Newton's law for one-dimensional motion: $md^2x/dt^2=F(x)$, a second-order differential equation, may be rewritten as a system of first-order equations:

$$dx/dt = v\ , \qquad m\, dv/dt = F(x) \tag{9}$$

Dividing these equations in order to eliminate dt, we have:

$$m\, v\, dv = F(x)\, dx = -\, dU$$

where

$$U(x) = -\int_0^x F(x')dx' \ \Leftrightarrow\ F(x) = -\, dU/dx\, .$$

Thus, $m\, v\, dv + dU = d\,(m\, v^2/2 + U) = 0 \ \Rightarrow$

$$m\, v^2/2 + U(x) = E \equiv const. \tag{10}$$

For each value of the constant E (total energy), Eq. (10) defines a curve in the 2-dimensional *phase space* with coordinates (x, v). This curve is called a *phase curve*. The value of E is uniquely determined by the initial conditions of the system, according to (5). Since the solution of the system (9) is unique for given initial conditions, *no two phase curves may intersect* in phase space. Let us see two examples:

1. Simple harmonic motion (cf. Sec. 3)

Conservation of mechanical energy in SHM is expressed by $mv^2/2 + kx^2/2 = E \ \Rightarrow$

$$\frac{x^2}{2E/k} + \frac{v^2}{2E/m} = 1 \qquad \text{(equation of an } \textit{ellipse}\text{)}$$

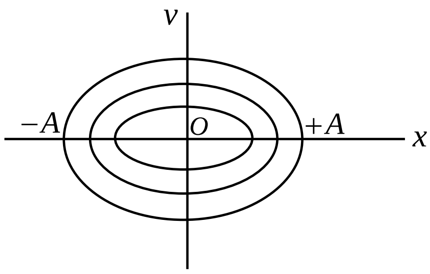

Fig. 2

Figure 2 shows a family of ellipses in phase space, corresponding to different values of E. Notice that, for $v=0 \Rightarrow x= \pm(2E/k)^{1/2} \equiv \pm A$, so that $E=kA^2/2$. Note also that the equations of motion, $\{dx/dt = v\ ,\ dv/dt = -kx/m\}$, endow the phase curves with a sense of direction for increasing t (i.e., for $dt > 0$). Indeed, the velocity v is positive (negative) for increasing (decreasing) x, while v decreases (increases) *algebraically* for positive (negative) x. This indicates that the phase curves are described *clockwise*.

2. Vertical motion under the force of gravity (cf. Sec. 4)

Conservation of mechanical energy is expressed by $mv^2/2 + mgx = E \ \Rightarrow$

$$v^2 = (2/m)\,(E - mgx) \qquad \text{(equation of a } parabola\text{)}$$

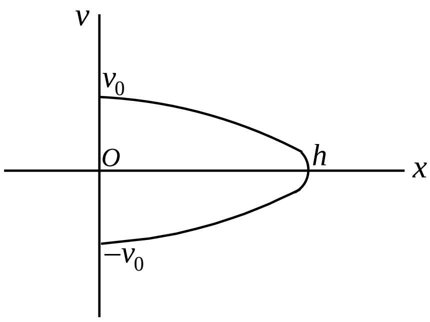

Fig. 3

Since $v^2 \geq 0$, we must have $E - mgx \geq 0 \Rightarrow x \leq E/mg$. Physically, this means that the particle will reach a maximum height $h=E/mg$ where it will stop momentarily and then its direction of motion will be reversed. On the other hand, at $x=0$ the velocity is $\pm v_0$ (see Fig. 3) where $v_0^2=2E/m \Rightarrow E=mv_0^2/2$. The maximum height is thus $h=v_0^2/2g$.

B. Oscillatory motion of (generally) non-conservative systems

1. Second-order linear differential equations

A second-order linear differential equation (DE) has the general form

$$y'' + a(x)\,y' + b(x)\,y = f(x) \tag{1}$$

where $y=y(x)$ and where $a(x)$, $b(x)$, $f(x)$ are given functions. If $f(x)\equiv 0$, the DE (1) is called *homogeneous linear* :

$$y'' + a(x)\,y' + b(x)\,y = 0 \tag{2}$$

As is easy to prove, if a function $y_1(x)$ is a solution of (2), then so is the function $y_2(x)=Cy_1(x)$ (C=const.). More generally, the following is true:

Theorem 1: If $y_1(x)$, $y_2(x)$,… are solutions of the homogeneous DE (2), then every linear combination of the form $y=C_1\,y_1(x)+C_2\,y_2(x)+\ldots$ (where C_1 , C_2 ,… are constants) also is a solution of (2).

Proof: By substituting for y on the left-hand side of (2) and by taking into account that each of the $y_1(x)$, $y_2(x)$,… satisfies this DE, we have:

$$y'' + a(x)\,y' + b(x)\,y = C_1\,(y_1'' + a\,y_1' + b\,y_1) + C_2\,(y_2'' + a\,y_2' + b\,y_2) + \ldots = 0\ .$$

Let $y_1(x)$ and $y_2(x)$ be two non-vanishing solutions of the homogeneous DE (2) [notice that the zero function $y(x)\equiv 0$ is a particular solution of (2)]. We say that the functions y_1 and y_2 are *linearly independent* if one is not a scalar multiple of the other. To put it in more formal terms, linear independence of y_1 and y_2 means that a relation of the form $C_1\,y_1(x)+C_2\,y_2(x)\equiv 0$ can only be true if $C_1=C_2=0$.

If we manage to find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous DE (2) (I can assure you that no other solution linearly independent of the former two exists!) then the *general solution* of (2) is the linear combination

$$y = C_1\,y_1(x) + C_2\,y_2(x) \tag{3}$$

where C_1 , C_2 are arbitrary constants.

Theorem 2: The general solution of the non-homogeneous DE (1) is the sum of the general solution (3) of the corresponding homogeneous equation (2) and *any particular solution* of (1).

Analytically: Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the homogeneous DE (2), and let $y_0(x)$ be any particular solution of (1). Then, the general solution of (1) is

$$y = C_1\,y_1(x) + C_2\,y_2(x) + y_0(x) \tag{4}$$

This practically means that, for any chosen y_0 , any other particular solution of (1) can be derived from (4) by properly choosing the constants C_1 and C_2 . Since (4) contains the totality of particular solutions of (1), it must be the general solution of (1).

2. Homogeneous linear equation with constant coefficients

This DE has the form

$$y'' + a y' + b y = 0 \qquad (5)$$

with constant a and b. It will be assumed that a and b are real numbers.

Theorem 3: If the complex function $y=u(x)+iv(x)$ satisfies the DE (5), then the same is true for each of the real functions $y_1=u(x)$ and $y_2=v(x)$ (real and imaginary part of y, respectively).

Proof: Putting $y=u+iv$ into (5), we find:

$$(u'' + a u' + b u) + i (v'' + a v' + b v) = 0 ,$$

which is true iff $u''+a u'+b u = 0$ and $v''+a v'+b v = 0$.

The standard method for solving (5) is the following: We try an exponential solution of the form $y=e^{kx}$. Then, $y'=ke^{kx}$, $y''=k^2e^{kx}$, and (5) yields (after eliminating e^{kx}):

$$k^2 + ak + b = 0 \quad (\textit{characteristic equation}) \qquad (6)$$

We distinguish the following cases:

1. Eq. (6) has real and distinct roots k_1 , k_2 . Then, the functions $e^{k_1 x}$ and $e^{k_2 x}$ are linearly independent and, according to (3), the general solution of (5) is of the form

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} \qquad (7)$$

2. Eq. (6) has real and equal roots, $k_1 = k_2 \equiv k$. The general solution of (5) is, in this case (check!),

$$y = (C_1 + C_2 x) e^{kx} \qquad (8)$$

3. Eq. (6) has complex conjugate roots $k_1=\alpha+i\beta$, $k_2=\alpha-i\beta$ (where α, β are real). The general solution of (5) is

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} = e^{\alpha x} (C_1 e^{i\beta x} + C_2 e^{-i\beta x}) .$$

By Euler's formula, $e^{\pm i\beta x} = \cos \beta x \pm i \sin \beta x$. We thus have:

$$y = e^{\alpha x} [(C_1 + C_2) \cos \beta x + i (C_1 - C_2) \sin \beta x] .$$

Since the (generally complex) constants C_1 and C_2 are arbitrary, we may put C_1 in place of C_1+C_2 and C_2 in place of $i (C_1 - C_2)$, so that, finally,

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) \qquad (9)$$

In any case, the general solution of (5) contains two arbitrary constants C_1 and C_2 . Upon assigning specific values to C_1 and C_2 we get a *particular solution* of (5). The values of C_1 and C_2 (and thus the particular solution itself) are determined from the general solution if we are given two *initial conditions* that the sought-for particular solution must obey. There are two kinds of initial conditions:

(a) We are given the values of $y(x)$ and $y'(x)$ for some value $x=x_0$ of x.

(b) We are given the values of $y(x)$ for $x=x_1$ and $x=x_2$.

Examples:

1. $y'' - y' - 2y = 0 \Rightarrow a = -1,\ b = -2$. The characteristic equation (6) is written: $k^2 - k - 2 = 0$, with real roots $k_1=2$, $k_2=-1$. The general solution (7) is $y = C_1 e^{2x} + C_2 e^{-x}$. Assume the initial conditions $y=2$ and $y'=-5$ when $x=0$. Then, $C_1=-1$, $C_2=3$ (show it!) and we get the *particular* solution $y = -e^{2x} + 3e^{-x}$.

2. $y'' - 6y' + 9y = 0 \Rightarrow a = -6,\ b=9$. The characteristic equation (6) is written: $k^2 - 6k + 9 = 0$, with real and equal roots $k_1=k_2=3$. The general solution (8) is $y = (C_1 + C_2 x)\, e^{3x}$.

3. $y'' - 4y' + 13y = 0 \Rightarrow a = -4,\ b=13$. The characteristic equation (6) is written: $k^2 - 4k + 13 = 0$, with complex conjugate roots $k_1=2+3i$, $k_2=2-3i$. The general solution (9) is (with $\alpha=2$, $\beta=3$): $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$. (Show that essentially the same result is found by making the alternative choice $\alpha=2$, $\beta=-3$.)

3. Harmonic oscillation

In a harmonic oscillation along the x-axis the total force on the oscillating body (of mass m) is $F=-kx$ ($k>0$), where x is the momentary displacement of the body from the position of equilibrium ($x=0$). By Newton's second law we have that $F=ma$, where a is the acceleration of the body: $a=d^2x/dt^2$. Therefore,

$$m\, d^2x/dt^2 = -kx$$

or, setting $k/m \equiv \omega^2$ (where we assume that $\omega>0$),

$$x'' + \omega^2 x = 0 \tag{10}$$

Eq. (10) is a homogeneous linear DE of the form (5) with x in place of y and t in place of x (notice that the first-derivative term is missing in this case). The characteristic equation (6) is written: $k^2+\omega^2=0$ (or, analytically, $k^2+0k+\omega^2=0$), with complex roots $k=\pm i\omega$ (analytically, $k_1=0+i\omega$, $k_2=0-i\omega$). The general solution of (10) is given by (9), with $\alpha=0$ and $\beta=\omega$:

$$x = C_1 \cos \omega t + C_2 \sin \omega t \tag{11}$$

where we assume that the constant coefficients C_1 and C_2 are real in order for the solution (11) to have physical meaning.

The general solution (11) can be put in different but equivalent form by setting

$$C_1 = A\sin\varphi\ ,\ \ C_2 = A\cos\varphi\ \ (A>0)\ \ \Leftrightarrow\ \ A=(C_1{}^2+C_2{}^2)^{1/2}\ ,\ \ \tan\varphi=C_1/C_2\ .$$

Then,

$$x = A\sin(\omega t+\varphi) \tag{12}$$

The positive constant A is called the *amplitude* of the oscillation, while the angle φ is called the *initial phase* (the value of the *phase* $\omega t+\varphi$ at time t=0). The positive constant ω is the *angular frequency* of oscillation, to be called just "*frequency*" in the sequel.

Notice that, if we set $C_1=A\cos\varphi$, $C_2=-A\sin\varphi$ in (11), we will get the general solution of (10) in the form

$$x = A\cos(\omega t+\varphi) \tag{13}$$

which is equivalent to (12). Indeed, equation (13) follows directly from (12) by putting $\varphi+(\pi/2)$ in place of φ (which is arbitrary anyway) in the latter equation.

4. Damped oscillation

In a damped oscillation, in addition to the restoring force $-kx$, opposite to the displacement x from the equilibrium position, there is a frictional force $-\lambda v= -\lambda dx/dt$ (λ>0) opposite to the velocity v. The total force on the body is[3] $F= -kx-\lambda dx/dt$. By Newton's law, $F=m\,d^2x/dt^2$. Hence,

$$m\,d^2x\,/\,dt^2 = -kx-\lambda\,dx/dt\ .$$

We set

$$k/m\equiv\omega_0{}^2\ \ (\omega_0= \textit{natural frequency of oscillation} \text{ without damping}),\quad \lambda/m\equiv 2\gamma,$$

so that

$$x'' + 2\gamma x' + \omega_0{}^2 x = 0 \tag{14}$$

Eq. (14) is a homogeneous linear DE. The characteristic equation (6) is

$$k^2 + 2\gamma k + \omega_0{}^2 = 0 \quad\Rightarrow\quad k = -\gamma \pm (\gamma^2-\omega_0{}^2)^{1/2}\ .$$

We distinguish the following cases:

1. *Large damping* $\Leftrightarrow$ $\gamma>\omega_0$. We have two real solutions:

$$k_1 = -\gamma + (\gamma^2-\omega_0{}^2)^{1/2}\ ,\quad k_2 = -\gamma - (\gamma^2-\omega_0{}^2)^{1/2}\ .$$

The general solution of (14) is of the form (7):

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t} \tag{15}$$

[3] Note that a velocity-dependent force is *not* conservative. Thus, conservation of energy methods do not apply in this case.

Let us assume that $C_1>0$ and $C_2>0$. Given that $k_1<0$ and $k_2<0$ (why?) we see that $x>0$ at all times t και, moreover, $x\to 0$ as $t\to\infty$. That is, as the time t increases, the moving object approaches the equilibrium position $x=0$ without ever crossing it. The motion is therefore *non-oscillatory*.

2. *Critical damping* $\Leftrightarrow$ $\gamma = \omega_0$. Then, $k_1 = k_2 = -\gamma$, and the general solution of (14) is of the form (8):

$$x = (C_1 + C_2\, t)\, e^{kt} = (C_1 + C_2\, t)\, e^{-\gamma t} \tag{16}$$

If we assume that $C_1>0$ and $C_2>0$, we see again that $x>0$ at all t and that $x\to 0$ as $t\to\infty$. (For the term $t\, e^{-\gamma t} = t \,/\, e^{\gamma t}$ we may use L'Hospital's rule for the indeterminate form ∞/∞; show this!) Thus, there is no oscillation in this case either.

3. *Small damping* $\Leftrightarrow$ $\gamma < \omega_0$. We have two complex conjugate solutions:

$$k = -\gamma \pm i\,\omega_1 \quad \text{where} \quad \omega_1 = (\omega_0{}^2 - \gamma^2)^{1/2} .$$

The general solution will be of the form (9), with $\alpha = -\gamma$ and $\beta = \omega_1$:

$$x = e^{-\gamma t} (C_1 \cos \omega_1 t + C_2 \sin \omega_1 t) ,$$

or, by setting $C_1 = A \sin\varphi$, $C_2 = A\cos\varphi$ ($A>0$),

$$x = A\, e^{-\gamma t} \sin(\omega_1 t + \varphi) \tag{17}$$

We notice that the amplitude $Ae^{-\gamma t}$ decreases exponentially with time (Fig. 1). Thus, strictly speaking, damped oscillatory motion is *not* periodic.

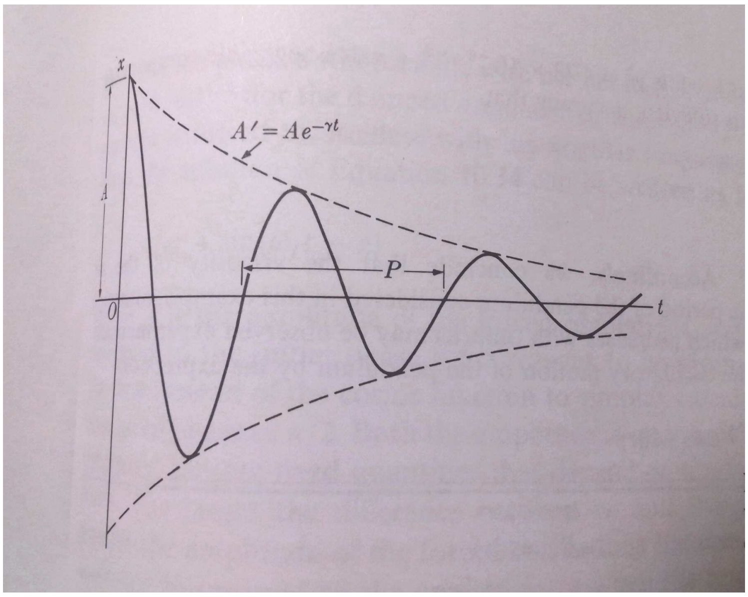

Fig. 1

5. Forced oscillation

In a forced oscillation, in addition to the restoring force $-kx$ and the frictional force $-\lambda v = -\lambda dx/dt$ the body is subject to an external force of the form

$$F(t) = F_0 \sin \omega_f t \quad (F_0 > 0) .$$

The total force on the body is $F = -kx - \lambda dx/dt + F_0 \sin \omega_f t$. By Newton's law we have that

$$m\, d^2x / dt^2 = -kx - \lambda\, dx/dt + F_0 \sin \omega_f t .$$

We set

$$k/m \equiv \omega_0^2 \;\; (\omega_0 = \textit{natural frequency}), \quad \lambda/m \equiv 2\gamma, \quad F_0/m \equiv f_0 ,$$

so that

$$x'' + 2\gamma x' + \omega_0^2 x = f_0 \sin \omega_f t \qquad (18)$$

Eq. (18) is a non-homogeneous linear DE. According to Theorem 2 of Sec. 1, its general solution is the sum of the general solution of the corresponding homogeneous equation,

$$x'' + 2\gamma x' + \omega_0^2 x = 0 ,$$

and *any particular solution* of (18). For small damping ($\gamma < \omega_0$) the general solution of the homogeneous equation is given by (17):

$$x = A_1 e^{-\gamma t} \sin(\omega_1 t + \varphi_1) \quad \text{where} \quad \omega_1 = (\omega_0^2 - \gamma^2)^{1/2} .$$

As can be verified, a particular solution of (18) is the following:

$$x = A \sin(\omega_f t + \varphi) \qquad (19)$$

where

$$A = \frac{f_0}{\left[\left(\omega_f^2 - \omega_0^2\right)^2 + 4\gamma^2 \omega_f^2\right]^{1/2}} \quad \text{and} \quad \tan\varphi = \frac{2\gamma\omega_f}{\omega_f^2 - \omega_0^2} \qquad (20)$$

The general solution of (18) is, therefore,

$$x = A_1 e^{-\gamma t} \sin(\omega_1 t + \varphi_1) + A \sin(\omega_f t + \varphi) \qquad (21)$$

with *arbitrary* A_1 , φ_1 . The first term on the right in (21) decreases exponentially with time and dies out quickly. In a steady-state situation, therefore, what remains is the particular solution (19):

$$x = A \sin(\omega_f t + \varphi) .$$

The amplitude A of oscillation is a function of the applied frequency ω_f, according to (20). This amplitude attains a maximum value when the denominator in the first relation (20) becomes minimum. This occurs when

$$\omega_f = (\omega_0^2 - 2\gamma^2)^{1/2} \equiv \omega_A \tag{22}$$

Proof: We set $\omega_f \equiv \omega$, for simplicity, and we consider the function

$$\Psi(\omega) = (\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2 ,$$

so that $A = f_0 / [\Psi(\omega)]^{1/2}$. We can show that

$$\Psi'(\omega) = 0 \text{ for } \omega = (\omega_0^2 - 2\gamma^2)^{1/2} = \omega_A \text{ and } \Psi''(\omega_A) = 8\omega_A^2 > 0 .$$

Thus, for small damping ($2\gamma^2 < \omega_0^2$) the function $\Psi(\omega)$ is *minimum*, hence the amplitude A is *maximum*, when $\omega_f = \omega_A$. This situation is called *amplitude resonance*.

In Fig. 2 it is assumed that $\lambda_1 < \lambda_2 \Leftrightarrow \gamma_1 < \gamma_2$. This means that, in accordance with (22), $\omega_{A,1} > \omega_{A,2}$. In the case of no damping ($\lambda=0 \Leftrightarrow \gamma=0$) Eq. (22) yields $\omega_A = \omega_0$. In other words, in an *undamped* forced oscillation the amplitude becomes maximum (in fact, infinite) when the applied frequency ω_f is equal to the natural frequency ω_0 of oscillation.

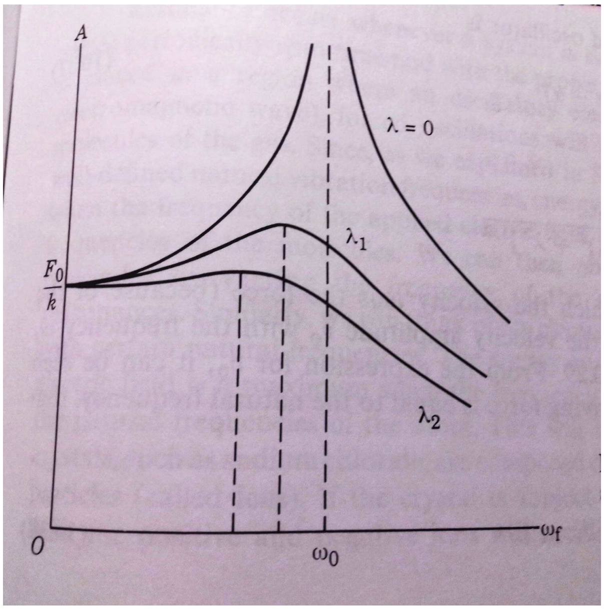

Fig. 2

By differentiating (19) we find the velocity of the oscillating body:

$$v = dx/dt = \omega_f A \cos(\omega_f t + \varphi) \equiv v_0 \cos(\omega_f t + \varphi)$$

where, by (20),

$$v_0 = \omega_f A = \frac{f_0}{\left[\left(1 - \frac{\omega_0^2}{\omega_f^2} \right)^2 + 4\gamma^2 \right]^{1/2}} \ .$$

The velocity amplitude v_0 becomes maximum when the denominator on the right is minimum, which occurs for $\omega_f = \omega_0$ (Fig. 3). The kinetic energy $mv_0^2/2$ then reaches its maximum value and there is *energy resonance*.

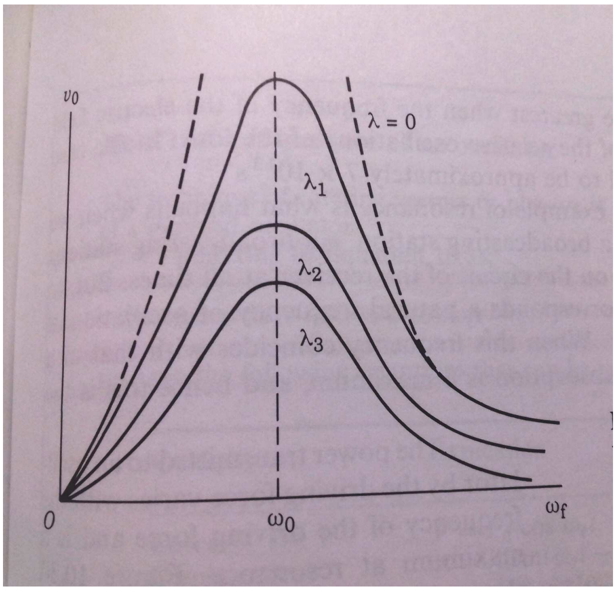

Fig. 3

Note that, in contrast to amplitude resonance, the frequency ω_f for energy resonance is independent of the damping factor λ and is always equal to the *natural frequency* ω_0 of the oscillator. At this frequency the work supplied by the external force $F(t)$ to the oscillator per unit time is maximum. That is, the oscillator absorbs the largest possible power from the external agent that exerts the force F.

Notice also that, in the case of zero damping ($\lambda=0 \Leftrightarrow \gamma=0$) the velocity amplitude v_0 becomes *infinite* at energy resonance, i.e., for $\omega_f = \omega_0$. This rather unphysical situation is, of course, purely theoretical since a mechanical motion with no friction whatsoever is practically impossible!

Appendix: Amplitude dependence of period

As we have shown, the general solution to the one-dimensional conservative Newtonian problem is

$$\int_{x_0}^{x} \frac{\pm\, dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = t - t_0 \tag{1}$$

where the plus sign is chosen for motion in the positive direction ($v>0$, $x>x_0$) while the minus sign applies to motion in the negative direction ($v<0$, $x<x_0$).

Let us assume that the potential energy $U(x)$ has the form of a U-shaped potential well (Fig. 1) such that $U(0)=0$ and $U(x)>0$ for $x\neq 0$. The graph of $U(x)$ is assumed to be symmetric with respect to the axis $x=0$, which means that $U(x)$ is an *even* function: $U(-x)=U(x)$.

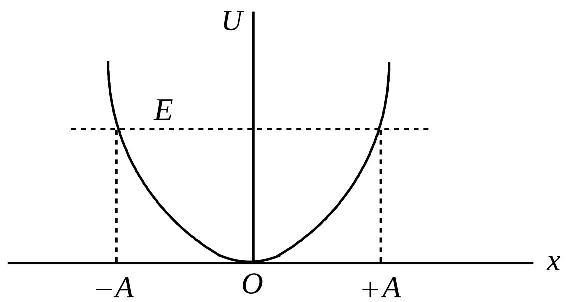

Fig. 1

If E is the total mechanical energy of the particle, then, according to Fig. 1, the motion is bounded between the points $-A$ and $+A$ of the x-axis, which are turning points at which the particle stops momentarily. Since E is constant, its value at all points equals its value at the turning points; i.e.,

$$E = U(\pm A) \tag{2}$$

The time it takes for a complete journey from $-A$ to $+A$ and back to $-A$ is found by using (1) with the appropriate sign for each direction of motion:

$$P = \int_{-A}^{A} \frac{dx}{\{\cdots\}^{1/2}} + \int_{A}^{-A} \frac{-dx}{\{\cdots\}^{1/2}} \;\Rightarrow$$

$$P = 2\int_{-A}^{A} \frac{dx}{\left\{\frac{2}{m}\left[E-U(x)\right]\right\}^{1/2}} = (2m)^{1/2} \int_{-A}^{A} \left[E-U(x)\right]^{-1/2} dx \tag{3}$$

Since P is fixed for a given A, the motion is periodic about the point $x=0$, with amplitude equal to A and with period P. It follows from (2) and (3) that the period P depends on A and thus on the total energy E of the particle. We will now show that an exception where P does *not* depend on A (thus on E also) is simple harmonic motion.

Since $U(x)$ is an even function with $U(0)=0$, it can be expanded into a Maclaurin series of the form

$$U(x) = \sum_{l=1}^{\infty} a_l \, x^{2l} \qquad (4)$$

where the coefficients a_l are not necessarily all different from zero. From (2) we have

$$E = U(\pm A) = \sum_{l=1}^{\infty} a_l \, A^{2l}$$

so that

$$E - U(x) = \sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right) .$$

Equation (3) then yields

$$P = \left(2m\right)^{1/2} \int_{-A}^{A} \left[\sum_{l=1}^{\infty} a_l \left(A^{2l} - x^{2l} \right) \right]^{-1/2} dx \, .$$

By setting $x/A=u \;\Leftrightarrow\; x=Au$, we get:

$$P = \left(2m\right)^{1/2} A \int_{-1}^{1} \left[\sum_{l=1}^{\infty} a_l \, A^{2l} \left(1 - u^{2l}\right) \right]^{-1/2} du \qquad (5)$$

It is obvious that, in general, P depends on A. The only exception where P is *not* dependent on A is the case where the following condition is satisfied: $a_l = 0$ for $l \neq 1$. That is, the only nonvanishing coefficient a_l in the series (4) is a_1. By setting $a_1 = k/2$ the potential energy (4) reduces to $U(x) = kx^2/2$, which corresponds to a restoring force of the form

$$F(x) = -\, dU/dx = -\, kx \qquad (6)$$

The periodic motion is then *simple harmonic motion* (SHM) and the period (5) reduces to

$$P = 2(m/k)^{1/2} \int_{-1}^{1} \left(1 - u^2\right)^{-1/2} du \;=\; 2(m/k)^{1/2} \left[\arcsin u\right]_{-1}^{1}$$

$$= 2(m/k)^{1/2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] \;\Rightarrow$$

$$P = 2\pi \left(\frac{m}{k} \right)^{1/2} \equiv \frac{2\pi}{\omega} \quad \text{where} \quad \omega = \frac{2\pi}{P} = \left(\frac{k}{m} \right)^{1/2} .$$

We notice that the period of SHM is amplitude-independent, hence also energy-independent.

It is of interest to examine a one-dimensional periodic motion that follows a *curved* path (where by "one-dimensional" we now mean that a single generalized co-ordinate – such as, e.g., an angle or a distance along the curve – is needed in order to specify the location of the particle). A nice example is that of an oscillating pendulum (Fig. 2; see also Sec. 5.5 and Problem 25 of [1]). The position of the mass m is specified by the arc length $OA=s=l\theta$ or, equivalently, by the angle θ (in rad). The algebraic value of the velocity of m is $v=ds/dt=l d\theta/dt$; it may be positive or negative, depending on the direction of motion relative to the unit tangent vector $\hat{u}_T$.

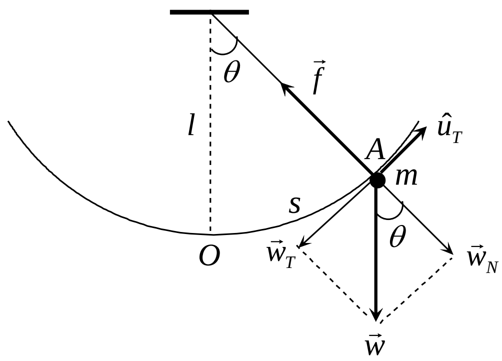

Fig. 2

The motion is governed by the tangential component $w_T = - mg \sin\theta$ (algebraic value) of the weight w. The tangential equation of motion of m is

$$m\, dv/dt = -mg \sin\theta \quad \Rightarrow \quad dv/dt = -g \sin\theta \tag{7}$$

We seek a conserved quantity that associates the velocity v with the position θ. We could, of course, work with (7) directly, but there is an easier way; namely, conservation of mechanical energy. This principle may be applied in view of the fact that the mass m is subject to the conservative force of gravity and the tension f of the string which, being normal to the velocity, produces no work (cf. Sec. 4.5 of [1]). The potential energy of m at point A (Fig. 2) is

$$U(\theta) = mg\,(l - l\cos\theta) = mgl\,(1-\cos\theta)\,,$$

where we have assumed that $U(0)=0$ (i.e., U is zero at the lowest point O). If α is the angular amplitude of oscillation (i.e., the maximum angle of deflection of the string from the vertical) then at $\theta= \pm\alpha$ the kinetic energy T vanishes and the total mechanical energy E is equal to $U(\pm\alpha)$. Applying conservation of mechanical energy between an arbitrary angle θ and the maximum angle $\theta=\alpha$, we have:

$$m v^2/2 + mgl\,(1-\cos\theta) = 0 + mgl\,(1-\cos\alpha) \;\Rightarrow\; \text{(after eliminating } m\text{)}$$

$$v^2 = 2gl\,(\cos\theta - \cos\alpha) \tag{8}$$

Exercise: By differentiating (8) with respect to t and by using the fact that $v=l d\theta/dt$, recover the equation of motion (7). Conversely, show that (8) is a direct consequence of (7). [*Hint:* Multiply (7) by v.]

Setting $v=ld\theta/dt$ in (8), we get a first-order differential equation:

$$d\theta/dt = \pm[(2g/l)(\cos\theta - \cos\alpha)]^{1/2} ,$$

which is integrated to give

$$\int_{\theta_0}^{\theta} \pm\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]^{-1/2} d\theta = t - t_0 \ .$$

The period of oscillation is [cf. Eq. (3)]

$$\begin{aligned} P &= 2\int_{-\alpha}^{\alpha}\left[\frac{2g}{l}(\cos\theta-\cos\alpha)\right]^{-1/2} d\theta \\ &= (2l/g)^{1/2}\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)^{-1/2}d\theta \end{aligned} \qquad (9)$$

Obviously, P depends on the angular amplitude α. Let us assume, however, that this amplitude is very small: $\alpha \ll 1$. We may then make the approximations

$$\cos\theta \approx 1 - \theta^2/2 \quad \text{and} \quad \cos\alpha \approx 1 - \alpha^2/2 \ .$$

Furthermore, we set $\theta/\alpha = u \Leftrightarrow \theta = \alpha u$. It is then a straightforward exercise to show that (9) reduces to

$$\begin{aligned} P &= 2(l/g)^{1/2}\int_{-1}^{1}\left(1-u^2\right)^{-1/2} du \;=\; 2(l/g)^{1/2}\left[\arcsin u\right]_{-1}^{1} \\ &= 2(l/g)^{1/2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] \;\Rightarrow \end{aligned}$$

$$P = 2\pi(l/g)^{1/2} ,$$

which is the familiar expression for the period of oscillation of a pendulum executing simple harmonic motion for small angles of deflection from the vertical. Once again, the SHM is seen to be the only one-dimensional periodic motion in which the period does not depend on the amplitude of oscillation.

Reference

[1] C. J. Papachristou, *Introduction to Mechanics of Particles and Systems* (Springer, 2020).[4]

[4] Manuscript: http://metapublishing.org/index.php/MP/catalog/book/68