Revisiting Archimedes' principle: Buoyancy and external pressure

Costas J. Papachristou

Department of Physical Sciences, Hellenic Naval Academy

A careful examination of Archimedes' principle shows that the buoyant force on a body that is either fully or partially immersed in a liquid is unaffected by the external (e.g. atmospheric) pressure, which acts both on the non-immersed part of the body (if any) and on the immersed part via Pascal's principle. The net force on the body due to the external pressure is zero and hence this pressure does not contribute to buoyancy.

1. A proper understanding of the buoyant force

In the statement of Archimedes' principle [1,2] we define the buoyant force on a body that is fully or partially immersed in a liquid as the resultant of all elementary normal forces exerted by the liquid on the immersed surface of the body. We then assert that this upward force is equal in magnitude to the weight of the fluid displaced by the immersed part of the body.

This definition of the buoyant force turns out to be consistent with Archimedes' principle for a body that is *fully immersed* (see Appendix). Moreover, the buoyant force is independent of the external pressure, which, by Pascal's principle, adds an extra constant pressure at all points of the surface of the immersed body. Indeed, as will be shown below, this extra pressure does not add any extra net force on the body. Thus, buoyancy is a hydrostatic effect due exclusively to the pressure exerted on the body by the liquid itself.

The situation is more delicate in the case of a (partially immersed) *floating* body. What we typically call "buoyant force" is the net force on the immersed surface of the body. By the equilibrium condition this force must be equal in magnitude to the weight of the body. But such a "balance" of forces makes no sense, given that the weight is a fixed force while the force on the immersed part may vary arbitrarily by changing the constant external pressure P_0 (again, this pressure is transferred to the immersed surface in accordance with Pascal's principle). To restore the balance of forces we must include the *downward* force on the *non-immersed* part of the body due to P_0. As it turns out, this force exactly matches the *upward* Pascal-oriented force on the immersed surface due to P_0 alone, so that, eventually, the force exerted over the *entire* surface of the body (both immersed and non-immersed) by the external pressure is zero. All we are left with, therefore, is the hydrostatic force due to the liquid alone, having magnitude equal to the weight of the displaced liquid. It is *this* force that will properly balance the weight of the body.

In conclusion: For consistency with Archimedes' principle, we must generally define the buoyant force on a body that is either fully or partially immersed in a liquid as the net force on the body due to the pressure exerted *by the liquid alone*, regardless of any (constant) external pressure. As shown below, the latter pressure contributes no additional net force on the body.

By properly defining the buoyant force, the balance of forces for a floating body, expressed by the equilibrium condition "buoyant force = total weight of the body", determines the percentage of the total volume of the body that is immersed in the liquid (cf. Sec. 8.9 of [1]). Since, as said above, the buoyant force is independent of the external pressure, it follows that we cannot make a floating body immerse further by increasing the external pressure on the free surface of the liquid!

2. Independence of buoyancy from external pressure

We propose to show that a *constant* external pressure P_0 does not affect the buoyant force on a body that is either fully or partially immersed in a liquid. [This pressure is felt directly on the non-immersed part (if any) as well as on the immersed part via Pascal's principle.] This means that an additional constant pressure over the *entire* surface of the body does not change the total force that would be exerted on the body by the liquid alone (i.e., if the external pressure P_0 did not exist). The net force on the entire surface of the body due to a constant external pressure P_0 must thus be zero.

Proposition: Consider a closed surface S placed inside a scalar field of constant value P_0 (Fig. 1). At each infinitesimal element ds of S the field exerts a force $d\vec{F}$ normal to ds and having magnitude dF proportional to the area of this surface element (which area will also be called ds): $dF=P_0\,ds$. We assume that, at each point of S, the elementary normal force $d\vec{F}$ is directed *toward* the surface, i.e., opposite to the local unit vector $\hat{n}$ normal to S and directed *outward*. Then, the total force exerted on S by the field P_0 is zero.

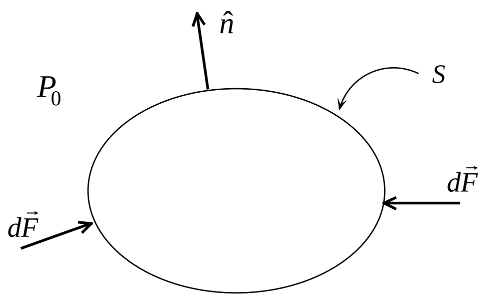


Fig. 1

Proof: We have that $d\vec{F} = -dF\,\hat{n} = -P_0\,ds\,\hat{n}$, so that the total force on S is

$$\vec{F} = \oint_S d\vec{F} = -P_0 \oint_S \hat{n}\,ds \qquad (1)$$

We show that, for any closed surface S, the following integral relation is true:

$$\vec{I} \equiv \oint_S \hat{n}\,ds = 0 \qquad (2)$$

It suffices to show that this vector relation is true when projected to *any* arbitrary direction. Let $\hat{b}$ be a unit vector defining such a direction. We write

$$I_b = \vec{I}\cdot\hat{b} = \oint_S \hat{b}\cdot\hat{n}\,ds\ .$$

Now, consider the *constant* vector field $\vec{F}(\vec{r})=\hat{b}$. Then, by using Gauss' integral theorem [3],

$$I_b=\oint_S \vec{F}(\vec{r})\cdot\hat{n}\,ds=\int_V(\vec{\nabla}\cdot\vec{F})\,dV=0 \quad [\text{since } \vec{\nabla}\cdot\vec{F}(\vec{r})=\vec{\nabla}\cdot\hat{b}=0\,]$$

where V is the volume enclosed by the surface S. Thus, the projection of the vector-valued integral $\vec{I}\equiv\oint_S \hat{n}\,ds$ to *any* arbitrary direction vanishes, which means that the vector relation (2) is true. Accordingly, the total force $\vec{F}$ on S, given by Eq. (1), is zero.

An alternative, more "intuitive" proof of the above Proposition is the following: Since S is a closed surface, for any unit vector $\hat{n}$ normal to S at some point of this surface there exists another point of S at which the normal unit vector is directed opposite to $\hat{n}$ (of course, both unit vectors are directed *outward* relative to the surface). This is easier to understand if instead of a closed surface we consider a closed plane curve C (see Fig. 2). If we make a full trip on C, the normal unit vector $\hat{n}$ will assume all possible directions until it finally returns to its original direction at the starting point of the trip. One of these (infinitely many) directions will be the opposite of the initial direction of $\hat{n}$.

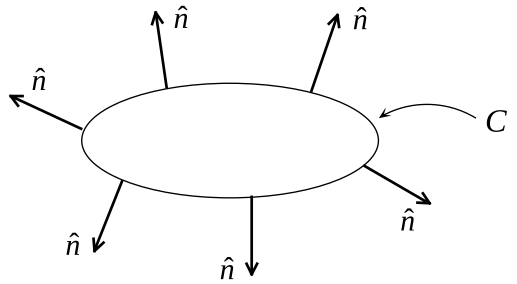


Fig. 2

Going back to our closed surface S, it follows from the above discussion that for every surface element $\hat{n}\,ds$ there is a corresponding element with opposite direction. This implies that $\oint_S \hat{n}\,ds=0$ (which is an interesting mathematical result in its own right). Hence, by Eq. (1), the total force $\vec{F}$ on S is zero. As seen in Fig. 3, for every elementary force $d\vec{F}$ on S there is always an opposite force $-d\vec{F}$ acting at some other point of the surface, so that, eventually, the net force on S by the constant field P_0 is zero.

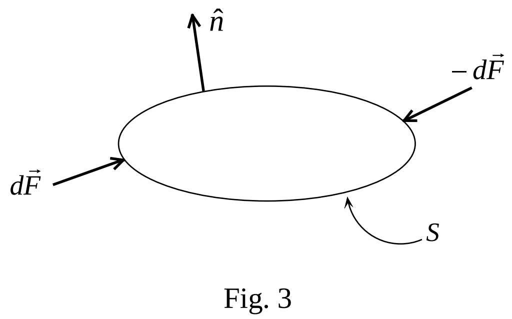
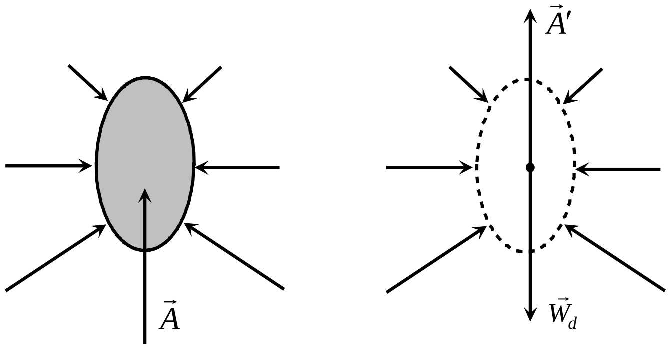


Fig. 3

In conclusion, a constant external pressure P_0 has no effect on the buoyant force experienced by a body that is either totally or partially immersed in a liquid. In particular, the equilibrium situation of a floating body will not be altered if we increase or decrease the external pressure.

Appendix: Proof of Archimedes' principle for a fully immersed body

For a fully immersed body the principle is proven theoretically as follows: Let us call V_d and $\vec{W}_d$ the volume and the weight, respectively, of the fluid displaced by the body. Since the body is fully immersed in the liquid, V_d equals the volume of the body.



(*a*) (*b*)

Fig. 4

Part (*a*) of Fig. 4 shows an instantaneous picture of the immersed body. The word "instantaneous" is related to the fact that, in general, the body is *not* in a state of equilibrium inside the liquid. The buoyant force $\vec{A}$ is typically defined as the resultant of all elementary forces acting normally on the surface of the body by the liquid.

In part (*b*) of Fig. 4 the body has been removed and has been replaced by liquid of the same volume and shape. The surface of that section of the fluid is now subject to a total force $\vec{A}'$ (buoyant force) from the surrounding fluid. The weight $\vec{W}_d$ of this fluid section is equal to the weight of the fluid that had previously been displaced by the body, while the line of action of $\vec{W}_d$ passes through the center of gravity of the displaced fluid.

In contrast to the submerged body, the part of the liquid that replaced the body is in a state of equilibrium since it is a portion of a fluid at rest. Hence,

$$\vec{A}' + \vec{W}_d = 0 \quad \Rightarrow \quad \vec{A}' = -\vec{W}_d \ .$$

Now, the buoyant force on the body is the same as the buoyant force on the part of the fluid replacing the body (i.e., $\vec{A} = \vec{A}'$) since the elementary forces exerted by a fluid on a surface are independent of the nature of the surface [1]. Thus, finally, the buoyant force exerted by the fluid on the body is $\vec{A} = -\vec{W}_d$. The direction of $\vec{A}$ is upward (i.e., opposite to the direction of $\vec{W}_d$) while its magnitude is $A = W_d = \rho g V_d$, where ρ is the density of the liquid.

We note that the total force $\vec{A}$ on the surface of the fully immersed body contains contributions from the constant external pressure P_0, which pressure is transferred via Pascal's principle to all points of the liquid. As we have shown, however, the net force due to P_0 over any closed surface (hence the surface of the body) is zero. Thus the buoyant force $\vec{A}$ is eventually *independent* of the external pressure P_0.

The case of a *partly* immersed floating body is more subtle, as we discussed earlier. The properly defined buoyant force is due to the pressure exerted on the immersed part of the body by the liquid alone, while the external pressure (acting on both immersed and non-immersed parts of the body) contributes no extra net force. Thus the buoyant force is independent of external pressure and equal in magnitude to the weight of the displaced fluid, in accordance with Archimedes' principle.

References

1. C. J. Papachristou, *Introduction to Mechanics of Particles and Systems* (Springer, 2020).
2. R. Resnick, D. Halliday, K. S. Krane, *Physics:* Volume 1, 5th Edition (Wiley, 2002).
3. M. D. Greenberg, *Advanced Engineering Mathematics,* 2nd Edition (Prentice-Hall, 1998).