# Galilean invariance of the work-energy theorem

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**Abstract:** The Galilean invariance of the work-energy theorem of Newtonian Mechanics is explicitly demonstrated.

*Definition:* A physical statement of Newtonian Mechanics is said to be *Galilean invariant* if it is valid with respect to all *inertial observers* (cf. Sec. 3.1 of [1]). If this statement is expressible by means of a mathematical equation, this equation must assume the *same form* in all *inertial frames of reference*.

Consider any two inertial observers $O$ and $O'$ with corresponding coordinate systems (or systems of axes) $(x, y, z)$ and $(x', y', z')$. Let $\vec{V}$ be the velocity of $O'$ relative to $O$. Clearly, this velocity is constant in time.

Consider also a particle of mass $m$, moving with velocity $\vec{v}$ and acceleration $\vec{a}$ with respect to $O$, and with velocity $\vec{v}'$ and acceleration $\vec{a}'$ with respect to $O'$. As shown in Sec. 2.8 of [1],

$$\begin{aligned} \vec{v}' &= \vec{v} - \vec{V} \\ \vec{a}' &= \vec{a} \end{aligned} \tag{1}$$

By Newton's 2nd law, the total force on $m$ according to $O$ and $O'$ is

$$\vec{F} = d\vec{p}/dt = m\vec{a} \quad \text{and} \quad \vec{F}' = d\vec{p}'/dt = m\vec{a}' ,$$

respectively, where $\vec{p} = m\vec{v}$ and $\vec{p}' = m\vec{v}'$. In view of (1), then,

$$\vec{F} = \vec{F}' \tag{2}$$

Assume now that the particle $m$ is inside a force field $\vec{F}(\vec{r})$ and moves from point $A$ to point $B$ along some curve in space. The inertial observers $O$ and $O'$ will generally perceive *different* trajectories of $m$ from $A$ to $B$. Both observers, however, define force according to Newton's 2nd law. Given that the work-energy theorem is a direct consequence of that law (see Sec. 4.3 of [1]), this theorem must be valid for both observers. That is, $W=\Delta E_k$ and, independently, $W'=\Delta E_k'$, where $W$ is the work done on $m$ by the field along the path $AB$, while $\Delta E_k = E_{k,B} - E_{k,A}$ is the change in the particle's kinetic energy along that path.

Let us now verify explicitly that, *if* $W=\Delta E_k$ for observer $O$, *then* $W'=\Delta E_k'$ for any other inertial observer $O'$.

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At time $t$ the particle $m$ passes through the trajectory point with position vector $\vec{r}(t)$ relative to observer $O$, or $\vec{r}'(t)$ relative to observer $O'$. By (2), both observers record the same force on $m$ at this instant, i.e.,

$$\vec{F}'\left(\vec{r}'(t)\right) = \vec{F}\left(\vec{r}(t)\right) \quad \text{or simply} \quad \vec{F}'(t) = \vec{F}(t) \tag{3}$$

(Careful: a prime does *not* denote a derivative with respect to $t$!) Now, let $W$ and $W'$ be the works done on $m$ from $A$ to $B$ according to $O$ and $O'$, respectively. We have:

$$W = \int_A^B \vec{F}(\vec{r}) \cdot d\vec{r} = \int_A^B \vec{F}\left(\vec{r}(t)\right) \cdot \frac{d\vec{r}}{dt} dt = \int_A^B \vec{F}(t) \cdot \vec{v}(t)\, dt$$

and, similarly,

$$W' = \int_A^B \vec{F}'(\vec{r}') \cdot d\vec{r}' = \int_A^B \vec{F}'(t) \cdot \vec{v}'(t)\, dt \,.$$

Taking (1) and (3) into account, we have:

$$W' = \int_A^B \vec{F}(t) \cdot \vec{v}(t)\, dt - \int_A^B \vec{F}(t) \cdot \vec{V} dt = W - \vec{V} \cdot \int_A^B \vec{F}(t)\, dt \,.$$

By using Newton's 2nd law, we have:

$$W' = W - m\vec{V} \cdot \int_A^B \frac{d\vec{v}}{dt} dt = W - m\vec{V} \cdot \int_A^B d\vec{v} \quad \Rightarrow$$

$$W' = W - m\vec{V} \cdot \left(\vec{v}_B - \vec{v}_A\right) \tag{4}$$

On the other hand, the change in kinetic energy from $A$ to $B$ is, according to $O$,

$$\Delta E_k = \frac{1}{2} m {v_B}^2 - \frac{1}{2} m {v_A}^2$$

while according to $O'$ and in view of (1),

$$\Delta E_k{}' = \frac{1}{2} m \left({v_B}'\right)^2 - \frac{1}{2} m \left({v_A}'\right)^2 \equiv \frac{1}{2} m \left(|\vec{v}_B{}'|^2 - |\vec{v}_A{}'|^2\right) = \frac{1}{2} m \left(|\vec{v}_B - \vec{V}|^2 - |\vec{v}_A - \vec{V}|^2\right).$$

By using the identity

$$|\vec{v} - \vec{V}|^2 = (\vec{v} - \vec{V}) \cdot (\vec{v} - \vec{V}) = v^2 + V^2 - 2\vec{v} \cdot \vec{V}$$

at $A$ and $B$, we find:

$$\Delta E_k{}' = \frac{1}{2} m \left( v_B{}^2 - v_A{}^2 - 2\vec{v}_B \cdot \vec{V} + 2\vec{v}_A \cdot \vec{V} \right) \Rightarrow$$

$$\Delta E_k{}' = \Delta E_k - m\vec{V} \cdot \left( \vec{v}_B - \vec{v}_A \right) \qquad (5)$$

Subtracting (5) from (4), we have: $W' - \Delta E_k' = W - \Delta E_k$ . So, *if* $W - \Delta E_k = 0 \Leftrightarrow W = \Delta E_k$ (i.e., if the work-energy theorem is valid in the *O*-frame) then $W' = \Delta E_k'$ (the theorem is valid in the *O*´-frame also). In other words, the work-energy theorem is Galilean invariant.

*Exercise:* Demonstrate in a similar way the Galilean invariance of the angular momentum – torque relation

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{T}$$

where $\vec{L} = m\vec{r} \times \vec{v}$ is the angular momentum of the particle *m* relative to *O*, and where $\vec{F}$ is the total force on *m* (see Sec. 3.7 of [1]).

[*Hint:* Assume that $\vec{r}' = \vec{r} - \vec{V}t$ (this means that the origins *O* and *O*´ of the two inertial frames coincide at *t*=0; as before, $\vec{V}$ is the constant velocity of *O*´ relative to *O*). Evaluate $\vec{L}' = m\vec{r}' \times \vec{v}'$ and, by using Newton's 2nd law, show that

$$\frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt} - t\vec{V} \times \vec{F} \qquad (6)$$

Also, show that $\vec{T}' = \vec{r}' \times \vec{F}'$ is equal to

$$\vec{T}' = \vec{T} - t\vec{V} \times \vec{F} \qquad (7)$$

Finally, subtract (7) from (6).]

## Reference


[1] C. J. Papachristou, *Introduction to Mechanics of Particles and Systems* (Springer, 2020), http://metapublishing.org/index.php/MP/catalog/book/68 .