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Bäcklund Transformations: Some Old and New Perspectives

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Abstract. Bäcklund transformations (BTs) are traditionally regarded as a tool for integrating nonlinear partial differential equations (PDEs). Their use has been recently extended, however, to problems such as the construction of recursion operators for symmetries of PDEs, as well as the solution of linear systems of PDEs. In this article, the concept and some applications of BTs are reviewed. As an example of an integrable linear system of PDEs, the Maxwell equations of electromagnetism are shown to constitute a BT connecting the wave equations for the electric and the magnetic field; plane-wave solutions of the Maxwell system are constructed in detail. The connection between BTs and recursion operators is also discussed.

Keywords: Bäcklund transformations, integrable systems, Maxwell equations, electromagnetic waves

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1. INTRODUCTION

Bäcklund transformations (BTs) were originally devised as a tool for obtaining solutions of nonlinear partial differential equations (PDEs) (see, e.g., [1] and the references therein). They were later also proven useful as *recursion operators* for constructing infinite sequences of nonlocal symmetries and conservation laws of certain PDEs [2–6].

In simple terms, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs [say, (*a*) and (*b*)] in order for the system to be integrable for either field. If a solution of PDE (*a*) is known, then a solution of PDE (*b*) is obtained simply by integrating the BT, without having to actually solve the latter PDE (which, presumably, would be a much harder task). In the case where the PDEs (*a*) and (*b*) are identical, the *auto-BT* produces new solutions of PDE (*a*) from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. But, what if the BT itself is the differential system whose solutions we are looking for? As it turns out, to solve the problem we need to have parameter-dependent solutions of *both* PDEs (*a*) and (*b*) at hand. By properly matching the parameters (provided this is possible) a solution of the given system is obtained.

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The above method is particularly effective in *linear* problems, given that parametric solutions of linear PDEs are generally not hard to find. An important paradigm of a BT associated with a linear problem is offered by the Maxwell system of equations of electromagnetism [7,8]. As is well known, the consistency of this system demands that both the electric and the magnetic field independently satisfy a respective wave equation. These equations have known, parameter-dependent solutions; namely, monochromatic plane waves with arbitrary amplitudes, frequencies and wave vectors (the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the appropriate expressions for the "parameters" in order for the plane waves to also be solutions of Maxwell's equations; that is, in order to represent an actual electromagnetic field.

This article, written for educational purposes, is an introduction to the concept of a BT and its application to the solution of PDEs or systems of PDEs. Both "classical" and novel views of a BT are discussed, the former view predominantly concerning integration of nonlinear PDEs while the latter one being applicable mostly to linear systems of PDEs. The article is organized as follows:

In Section 2 we review the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 a different perception of a BT is presented, according to which it is the BT itself whose solutions are sought. The concept of *conjugate solutions* is introduced.

As an example, in Secs. 4 and 5 the Maxwell equations in empty space and in a linear conducting medium, respectively, are shown to constitute a BT connecting the wave equations for the electric and the magnetic field. Following [7], the process of constructing plane-wave solutions of this BT is presented in detail. This process is, of course, a familiar problem of electrodynamics but is seen here under a new perspective by employing the concept of a BT.

Finally, in Sec. 6 we briefly review the connection between BTs and recursion operators for generating infinite sequences of nonlocal symmetries of PDEs.

2. BÄCKLUND TRANSFORMATIONS: CLASSICAL VIEWPOINT

Consider two PDEs $P[u]=0$ and $Q[v]=0$ for the unknown functions u and v, respectively. The expressions $P[u]$ and $Q[v]$ may contain the corresponding variables u and v, as well as partial derivatives of u and v with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, t. Partial derivatives with respect to these variables will be denoted by using subscripts: u_x, u_t, u_{xx}, u_{tt}, u_{xt}, etc.

Independently, for the moment, also consider a pair of coupled PDEs for u and v:

$$B_1[u,v]=0 \quad (a) \qquad B_2[u,v]=0 \quad (b) \tag{1}$$

where the expressions $B_i[u,v]$ (i=1,2) may contain u, v as well as partial derivatives of u and v with respect to x and t. We note that u appears in both equations (a) and (b). The question then is: if we find an expression for u by integrating (a) for a given v, will it match the corresponding expression for u found by integrating (b) for the same v? The answer is that, in order that (a) and (b) be consistent with each other for solution for u, the function v must be properly chosen so as to satisfy a certain *consistency condition* (or *integrability condition* or *compatibility condition*).

By a similar reasoning, in order that (a) and (b) in (1) be mutually consistent for solution for v, for some given u, the function u must now itself satisfy a corresponding integrability condition.

If it happens that the two consistency conditions for integrability of the system (1) are precisely the PDEs $P[u]=0$ and $Q[v]=0$, we say that the above system constitutes a *Bäcklund* *transformation* (BT) connecting solutions of $P[u]=0$ with solutions of $Q[v]=0$. In the special case where $P\equiv Q$, i.e., when u and v satisfy *the same* PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT) for this PDE.

Suppose now that we seek solutions of the PDE $P[u]=0$. Assume that we are able to find a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$, the auto-BT connects solutions u and v of the same PDE) and let $v=v_0(x,t)$ be some known solution of $Q[v]=0$. The BT is then a system of PDEs for the unknown u,

$$B_i[u,v_0]=0\,,\quad i=1,2 \tag{2}$$

The system (2) is integrable for u, given that the function v_0 satisfies *a priori* the required integrability condition $Q[v]=0$. The solution u then of the system satisfies the PDE $P[u]=0$. Thus a solution $u(x,t)$ of the latter PDE is found without actually solving the equation itself, simply by integrating the BT (2) with respect to u. Of course, this method will be useful provided that integrating the system (2) for u is simpler than integrating the PDE $P[u]=0$ itself. If the transformation (2) is an auto-BT for the PDE $P[u]=0$, then, starting with a known solution $v_0(x,t)$ of this equation and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of the use of a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of Complex Analysis,

$$u_x=v_y \quad (a) \qquad u_y=-v_x \quad (b) \tag{3}$$

(here, the variable t has been renamed y) constitute an auto-BT for the *Laplace equation*,

$$P[w]\equiv w_{xx}+w_{yy}=0 \tag{4}$$

Let us explain this: Suppose we want to solve the system (3) for u, for a given choice of the function $v(x,y)$. To see if the PDEs (a) and (b) match for solution for u, we must compare them in some way. We thus differentiate (a) with respect to y and (b) with respect to x, and equate the mixed derivatives of u. That is, we apply the integrability condition $(u_x)_y=(u_y)_x$. In this way we eliminate the variable u and find the condition that must be obeyed by $v(x,y)$:

$$P[v]\equiv v_{xx}+v_{yy}=0\ .$$

Similarly, by using the integrability condition $(v_x)_y=(v_y)_x$ to eliminate v from the system (3), we find the necessary condition in order that this system be integrable for v, for a given function $u(x,y)$:

$$P[u]\equiv u_{xx}+u_{yy}=0\ .$$

In conclusion, the integrability of system (3) with respect to either variable requires that the other variable must satisfy the Laplace equation (4).

Let now $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate this system with respect to u. It is not hard to show (by eliminating v_0 from the system) that the solution u will also satisfy the Laplace equation (4). As an example, by choosing the solution $v_0(x,y)=xy$, we find a new solution $u(x,y)=(x^2-y^2)/2+C$.

2. The *Liouville equation* is written

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$$P[u] \equiv u_{xt} - e^u = 0 \quad \Leftrightarrow \quad u_{xt} = e^u \tag{5}$$

Due to its nonlinearity, this PDE is hard to integrate directly. A solution is thus sought by means of a BT. We consider an auxiliary function $v(x,t)$ and an associated PDE,

$$Q[v] \equiv v_{xt} = 0 \tag{6}$$

We also consider the system of first-order PDEs,

$$u_x + v_x = \sqrt{2}\, e^{(u-v)/2} \quad (a) \qquad u_t - v_t = \sqrt{2}\, e^{(u+v)/2} \quad (b) \tag{7}$$

Differentiating the PDE (a) with respect to t and the PDE (b) with respect to x, and eliminating $(u_t - v_t)$ and $(u_x + v_x)$ in the ensuing equations with the aid of (a) and (b), we find that u and v satisfy the PDEs (5) and (6), respectively. Thus, the system (7) is a BT connecting solutions of (5) and (6). Starting with the trivial solution v=0 of (6), and integrating the system

$$u_x = \sqrt{2}\, e^{u/2} \;, \qquad u_t = \sqrt{2}\, e^{u/2} \;,$$

we find a nontrivial solution of (5):

$$u(x,t) = -2 \ln\left(C - \frac{x+t}{\sqrt{2}} \right) .$$

3. The "*sine-Gordon" equation* has applications in various areas of Physics, e.g., in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name is a pun on the related linear Klein-Gordon equation) is written

$$P[u] \equiv u_{xt} - \sin u = 0 \quad \Leftrightarrow \quad u_{xt} = \sin u \tag{8}$$

The following system of equations is an auto-BT for the nonlinear PDE (8):

$$\frac{1}{2}(u+v)_x = a \sin\left(\frac{u-v}{2}\right) \;, \qquad \frac{1}{2}(u-v)_t = \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \tag{9}$$

where $a\,(\neq 0)$ is an arbitrary real constant. [Because of the presence of a, the system (9) is called a *parametric* BT.] When u is a solution of (8) the BT (9) is integrable for v, which, in turn, also is a solution of (8): $P[v]$=0; and vice versa. Starting with the trivial solution v=0 of $v_{xt}= \sin v$, and integrating the system

$$u_x = 2a \sin\frac{u}{2} \;, \qquad u_t = \frac{2}{a} \sin\frac{u}{2} \;,$$

we obtain a new solution of (8):

$$u(x,t) = 4 \arctan\left\{ C \exp\left(ax + \frac{t}{a} \right) \right\} .$$

3. CONJUGATE SOLUTIONS AND ANOTHER VIEW OF A BT

As presented in the previous section, a BT is an auxiliary device for constructing solutions of a (usually nonlinear) PDE from known solutions of the same or another PDE. The converse problem, where solutions of the differential system representing the BT itself are sought, is also of interest, however, and has been recently suggested [7,8] in connection with the Maxwell equations (see subsequent sections).

To be specific, assume that we need to integrate a given system of PDEs connecting two functions *u* and *v*:

$$B_i\,[u,v]=0\ ,\quad i=1,2 \tag{10}$$

Suppose that the integrability of the system for both functions requires that *u* and *v* separately satisfy the respective PDEs

$$P[u]=0\quad (a)\qquad Q[v]=0\quad (b) \tag{11}$$

That is, the system (10) is a BT connecting solutions of the PDEs (11). Assume, now, that these PDEs possess known (or, in any case, easy to find) *parameter-dependent solutions* of the form

$$u=f\,(x,y\,;\alpha,\beta,\ldots)\ ,\qquad v=g(x,y\,;\kappa,\lambda,\ldots) \tag{12}$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which *u* and *v* jointly satisfy the system (10), we say that the solutions *u* and *v* of the PDEs (11*a*) and (11*b*), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short). By finding a pair of BT-conjugate solutions one thus automatically obtains a solution of the system (10).

Note that solutions of *both* integrability conditions $P[u]=0$ and $Q[v]=0$ must now be known in advance! From the practical point of view the method is thus most applicable in *linear* problems, since it is much easier to find parameter-dependent solutions of the PDEs (11) in this case.

Let us see an example: Going back to the Cauchy-Riemann relations (3), we try the following parametric solutions of the Laplace equation (4):

$$u\,(x,y)=\alpha\,(x^2-y^2)+\beta x+\gamma y\ ,$$
$$v\,(x,y)=\kappa xy+\lambda x+\mu y\ .$$

Substituting these into the BT (3), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$u\,(x,y)=\alpha\,(x^2-y^2)+\beta x+\gamma y\ ,$$
$$v\,(x,y)=2\alpha xy-\gamma x+\beta y$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u\,(x,y)=\alpha xy\ ,\quad v\,(x,y)=\beta xy\ .$$

Inserting these into the system (3) and taking into account the independence of *x* and *y*, we find that the only possible values of the parameters *α* and *β* are *α*=*β*=0, so that *u*(*x*,*y*)= *v*(*x*,*y*)=0. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. EXAMPLE: THE MAXWELL EQUATIONS IN EMPTY SPACE

An example of an integrable linear system whose solutions are of physical interest is furnished by the *Maxwell equations* of electrodynamics. Interestingly, as noted recently [7], the Maxwell system has the property of a BT whose integrability conditions are the electromagnetic (e/m) wave equations that are separately valid for the electric and the magnetic field. These equations possess parameter-dependent solutions that, by a proper choice of the parameters, can be made BT-conjugate through the Maxwell system. In this and the following section we discuss the BT property of the Maxwell equations in vacuum and in a conducting medium, respectively.

In *empty space*, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units) [9]

$$\begin{aligned} &(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad &(c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad &(d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{13}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. Here we have a system of four PDEs for two fields. The question is: what are the necessary conditions that each of these fields must satisfy in order for the system (13) to be self-consistent? In other words, what are the *consistency conditions* (or *integrability conditions*) for this system?

Guided by our experience from Sec. 2, to find these conditions we perform various differentiations of the equations of system (13) and require that certain differential identities be satisfied. Our aim is, of course, to eliminate one field (electric or magnetic) in favor of the other and find some higher-order PDE that the latter field must obey.

As can be checked, two differential identities are satisfied automatically in the system (13):

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})=0\ ,\quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})=0\ ,$$

$$(\vec{\nabla}\cdot\vec{E})_t=\vec{\nabla}\cdot\vec{E}_t\ ,\quad (\vec{\nabla}\cdot\vec{B})_t=\vec{\nabla}\cdot\vec{B}_t\ .$$

Two others read

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{14}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{15}$$

Taking the *rot* of (13*c*) and using (14), (13*a*) and (13*d*), we find

$$\nabla^2 \vec{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad (16)$$

Similarly, taking the *rot* of (13*d*) and using (15), (13*b*) and (13*c*), we get

$$\nabla^2 \vec{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \qquad (17)$$

No new information is furnished by the remaining two integrability conditions,

$$(\vec{\nabla} \times \vec{E})_t = \vec{\nabla} \times \vec{E}_t \quad , \quad (\vec{\nabla} \times \vec{B})_t = \vec{\nabla} \times \vec{B}_t \quad .$$

Note that we have *uncoupled* the equations for the two fields in the system (13), deriving separate second-order PDEs for each field. Putting

$$\varepsilon_0 \mu_0 \equiv \frac{1}{c^2} \quad \Leftrightarrow \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad (18)$$

(where *c* is the speed of light in vacuum) we rewrite (16) and (17) in wave-equation form:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad (19)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \qquad (20)$$

We conclude that the Maxwell system (13) is a BT relating solutions of the e/m wave equations (19) and (20), these equations representing the integrability conditions of the BT. It should be noted that this BT is *not* an *auto*-BT! Indeed, although the PDEs (19) and (20) are of similar form, they concern *different* fields with different physical dimensions and physical properties.

The e/m wave equations admit plane-wave solutions of the form $\vec{F}(\vec{k} \cdot \vec{r} - \omega t)$, with

$$\frac{\omega}{k} = c \quad \text{where} \quad k = |\vec{k}| \qquad (21)$$

The simplest such solutions are *monochromatic plane waves* of angular frequency *ω*, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} \qquad (a) \\ \vec{B}(\vec{r},t) &= \vec{B}_0 \exp\{i(\vec{k} \cdot \vec{r} - \omega t)\} \qquad (b) \end{aligned} \qquad (22)$$

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where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes. The constants appearing in the above equations (amplitudes, frequency and wave vector) can be chosen arbitrarily; thus they can be regarded as *parameters* on which the plane waves (22) depend.

We must note carefully that, although every pair of fields $(\vec{E},\vec{B})$ satisfying the Maxwell equations (13) also satisfies the wave equations (19) and (20), the converse is not true. Thus, the plane-wave solutions (22) are not *a priori* solutions of the Maxwell system (i.e., do not represent actual e/m fields). This problem can be taken care of, however, by a proper choice of the parameters in (22). To this end, we substitute the general solutions (22) into the BT (13) to find the extra conditions the latter system demands. By fixing the wave parameters, the two wave solutions in (22) will become *BT-conjugate* through the Maxwell system (13).

Substituting (22*a*) and (22*b*) into (13*a*) and (13*b*), respectively, and taking into account that $\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$, we have

$$(\vec{E}_0\, e^{-i\omega t})\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = 0 \;\Rightarrow\; (\vec{k}\cdot\vec{E}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0\,,$$
$$(\vec{B}_0\, e^{-i\omega t})\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = 0 \;\Rightarrow\; (\vec{k}\cdot\vec{B}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0\,,$$

so that

$$\vec{k}\cdot\vec{E}_0 = 0\,, \quad \vec{k}\cdot\vec{B}_0 = 0\,. \tag{23}$$

Relations (23) reflect the fact that that the monochromatic plane e/m wave is a *transverse wave*.

Next, substituting (22*a*) and (22*b*) into (13*c*) and (13*d*), we find

$$e^{-i\omega t}(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}})\times\vec{E}_0 = i\,\omega\,\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{E}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \omega\,\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,$$

$$e^{-i\omega t}(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}})\times\vec{B}_0 = -i\,\omega\,\varepsilon_0\mu_0\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{B}_0)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\frac{\omega}{c^2}\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,,$$

so that

$$\vec{k}\times\vec{E}_0 = \omega\,\vec{B}_0\,, \quad \vec{k}\times\vec{B}_0 = -\frac{\omega}{c^2}\vec{E}_0 \tag{24}$$

We note that the fields $\vec{E}$ and $\vec{B}$ are normal to each other, as well as normal to the direction of propagation of the wave. We also remark that the two vector equations in (24) are not independent of each other, since, by cross-multiplying the first relation by $\vec{k}$, we get the second relation.

Introducing a unit vector $\hat{\tau}$ in the direction of the wave vector $\vec{k}$,

$$\hat{\tau} = \vec{k}/k \quad (k = |\vec{k}| = \omega/c)\,,$$

we rewrite the first of equations (24) as

$$\vec{B}_0 = \frac{k}{\omega}(\hat{\tau}\times\vec{E}_0) = \frac{1}{c}(\hat{\tau}\times\vec{E}_0) \ .$$

The BT-conjugate solutions in (22) are now written

$$\begin{aligned} &\vec{E}(\vec{r},t) = \vec{E}_0 \exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \ , \\ &\vec{B}(\vec{r},t) = \frac{1}{c}(\hat{\tau}\times\vec{E}_0)\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} = \frac{1}{c}\,\hat{\tau}\times\vec{E} \end{aligned} \tag{25}$$

As constructed, the complex vector fields in (25) satisfy the Maxwell system (13). Since this system is homogeneous linear with real coefficients, the real parts of the fields (25) also satisfy it. To find the expressions for the real solutions (which, after all, carry the physics of the situation) we take the simplest case of *linear polarization* and write

$$\vec{E}_0 = \vec{E}_{0,R}\, e^{i\alpha} \tag{26}$$

where the vector $\vec{E}_{0,R}$ as well as the number α are real. The *real* versions of the fields (25), then, read

$$\begin{aligned} &\vec{E} = \vec{E}_{0,R}\cos(\vec{k}\cdot\vec{r} - \omega t + \alpha) \ , \\ &\vec{B} = \frac{1}{c}(\hat{\tau}\times\vec{E}_{0,R})\cos(\vec{k}\cdot\vec{r} - \omega t + \alpha) = \frac{1}{c}\,\hat{\tau}\times\vec{E} \end{aligned} \tag{27}$$

We note, in particular, that the fields $\vec{E}$ and $\vec{B}$ "oscillate" in phase.

Our results for the Maxwell equations in vacuum can be extended to the case of a *linear non-conducting medium* upon replacement of ε_0 and μ_0 with ε and μ, respectively. The speed of propagation of the e/m wave is, in this case,

$$\upsilon = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \ .$$

In the next section we study the more complex case of a linear medium having a finite conductivity.

5. EXAMPLE: THE MAXWELL SYSTEM FOR A LINEAR CONDUCTING MEDIUM

Consider a linear conducting medium of conductivity σ. In such a medium, Ohm's law is satisfied: $\vec{J}_f = \sigma\vec{E}$, where $\vec{J}_f$ is the free current density. The Maxwell equations take on the form [9]

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$$\begin{aligned} &(a) \quad \vec{\nabla}\cdot\vec{E}=0 && (c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &(b) \quad \vec{\nabla}\cdot\vec{B}=0 && (d) \quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{28}$$

By requiring satisfaction of the integrability conditions

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \ ,$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \ ,$$

we obtain the *modified wave equations*

$$\begin{aligned} &\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \\ &\nabla^2\vec{B}-\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\frac{\partial\vec{B}}{\partial t}=0 \end{aligned} \tag{29}$$

which must be separately satisfied by each field. As in Sec. 4, no further information is furnished by the remaining integrability conditions.

The linear differential system (28) is a BT relating solutions of the wave equations (29). As in the vacuum case, this BT is *not* an auto-BT. We now seek BT-conjugate solutions. As can be verified by direct substitution into equations (29), these PDEs admit parameter-dependent solutions of the form

$$\begin{aligned} \vec{E}(\vec{r},t)&=\vec{E}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\} \\ &=\vec{E}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t) \ , \\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\} \\ &=\vec{B}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t) \end{aligned} \tag{30}$$

where $\hat{\tau}$ is the unit vector in the direction of the wave vector $\vec{k}$:

$$\hat{\tau}=\vec{k}/k \quad (k=|\vec{k}|=\omega/\upsilon)$$

(υ is the speed of propagation of the wave inside the conducting medium) and where, for given physical characteristics ε, μ, σ of the medium, the parameters s, k and ω satisfy the algebraic system

$$s^2-k^2+\varepsilon\mu\omega^2=0 \ , \quad \mu\sigma\omega-2sk=0 \tag{31}$$

We note that, for arbitrary choices of the amplitudes $\vec{E}_0$ and $\vec{B}_0$, the vector fields (30) are not *a priori* solutions of the Maxwell system (28), thus are not BT-conjugate solutions. To obtain such solutions we substitute expressions (30) into the system (28). With the aid of the relation

$$\vec{\nabla} e^{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}} = \left(i-\frac{s}{k}\right)\vec{k}\, e^{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}}$$

one can show that (28*a*) and (28*b*) impose the conditions

$$\vec{k}\cdot\vec{E}_0 = 0\,, \quad \vec{k}\cdot\vec{B}_0 = 0 \tag{32}$$

As in the vacuum case, the e/m wave in a conducting medium is a *transverse* wave.

By substituting (30) into (28*c*) and (28*d*), two more conditions are found:

$$(k+is)\,\hat{\tau}\times\vec{E}_0 = \omega\vec{B}_0 \tag{33}$$

$$(k+is)\,\hat{\tau}\times\vec{B}_0 = -(\varepsilon\mu\omega + i\mu\sigma)\vec{E}_0 \tag{34}$$

Note, however, that (34) is not an independent equation since it can be reproduced by cross-multiplying (33) by $\hat{\tau}$, taking into account the algebraic relations (31).

The BT-conjugate solutions of the wave equations (29) are now written

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,, \\ \vec{B}(\vec{r},t) &= \frac{k+is}{\omega}\,(\hat{\tau}\times\vec{E}_0)\, e^{-s\,\hat{\tau}\cdot\vec{r}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{aligned} \tag{35}$$

To find the corresponding real solutions, we assume linear polarization of the wave, as before, and set

$$\vec{E}_0 = \vec{E}_{0,R}\, e^{i\alpha}\,.$$

We also put

$$k+is = |\,k+is\,|\,e^{i\varphi} = \sqrt{k^2+s^2}\;e^{i\varphi} \;\;;\;\; \tan\varphi = s/k\,.$$

Taking the real parts of equations (35), we finally have:

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_{0,R}\, e^{-s\,\hat{\tau}\cdot\vec{r}} \cos(\vec{k}\cdot\vec{r}-\omega t+\alpha)\,, \\ \vec{B}(\vec{r},t) &= \frac{\sqrt{k^2+s^2}}{\omega}\,(\hat{\tau}\times\vec{E}_{0,R})\, e^{-s\,\hat{\tau}\cdot\vec{r}} \cos(\vec{k}\cdot\vec{r}-\omega t+\alpha+\varphi)\,. \end{aligned}$$

As an exercise, the student may show that these results reduce to those for a linear non-conducting medium (cf. Sec. 4) in the limit $\sigma \rightarrow 0$.

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6. BTS AS RECURSION OPERATORS

The concept of symmetries of PDEs was discussed in [1]. Let us review the main facts:

Consider a PDE $F[u]=0$, where, for simplicity, $u=u(x,t)$. A transformation

$$u(x,t) \rightarrow u'(x,t)$$

from the function u to a new function u' represents a *symmetry* of the given PDE if the following condition is satisfied: $u'(x,t)$ is a solution of $F[u]=0$ *if* $u(x,t)$ is a solution. That is,

$$F[u']=0 \quad when \quad F[u]=0 \tag{36}$$

An *infinitesimal symmetry transformation* is written

$$u' = u+\delta u = u+\alpha Q[u] \tag{37}$$

where α is an infinitesimal parameter. The function $Q[u] \equiv Q(x, t, u, u_x, u_t, ...)$ is called the *symmetry characteristic* of the transformation (37).

In order that a function $Q[u]$ be a symmetry characteristic for the PDE $F[u]=0$, it must satisfy a certain PDE that expresses the *symmetry condition* for $F[u]=0$. We write, symbolically,

$$S(Q;u)=0 \quad when \quad F[u]=0 \tag{38}$$

where the expression S depends *linearly* on Q and its partial derivatives. Thus, (38) is a linear PDE for Q, in which equation the variable u enters as a sort of parametric function that is required to satisfy the PDE $F[u]=0$.

A *recursion operator* $\hat{R}$ [10] is a linear operator which, acting on a symmetry characteristic Q, produces a new symmetry characteristic $Q' = \hat{R}Q$. That is,

$$S(\hat{R}Q;u)=0 \quad when \quad S(Q;u)=0 \tag{39}$$

It is not too difficult to show that *any power of a recursion operator also is a recursion operator*. This means that, starting with any symmetry characteristic Q, one may in principle obtain an infinite set of characteristics (thus, an infinite number of symmetries) by repeated application of the recursion operator.

A new approach to recursion operators was suggested in the early 1990s [2,3] (see also [4-6]). According to this view, a recursion operator is an auto-BT for the linear PDE (38) expressing the symmetry condition of the problem; that is, a BT producing new solutions Q' of (38) from old ones, Q. Typically, this type of BT produces *nonlocal* symmetries, i.e., symmetry characteristics depending on *integrals* (rather than derivatives) of u.

As an example, consider the *chiral field equation*

$$F[g] \equiv (g^{-1}g_x)_x + (g^{-1}g_t)_t = 0 \tag{40}$$

(as usual, subscripts denote partial differentiations) where g is a $GL(n,C)$-valued function of x and t (i.e., an invertible complex $n \times n$ matrix, differentiable for all x, t).

Let $Q[g]$ be a symmetry characteristic of the PDE (40). It is convenient to put

$$Q[g] = g\,\Phi[g]$$

and write the corresponding infinitesimal symmetry transformation in the form

$$g' = g + \delta g = g + \alpha g \Phi[g] \tag{41}$$

The symmetry condition that Q must satisfy will be a PDE linear in Q, thus in Φ also. As can be shown [4], this PDE is

$$S(\Phi; g) \equiv \Phi_{xx} + \Phi_{tt} + [g^{-1}g_x, \Phi_x] + [g^{-1}g_t, \Phi_t] = 0 \tag{42}$$

which must be valid when $F[g]=0$ (where, in general, $[A, B] \equiv AB-BA$ denotes the *commutator* of two matrices A and B).

For a given g satisfying $F[g]=0$, consider now the following system of PDEs for the matrix functions Φ and Φ':

$$\begin{aligned} \Phi'_x &= \Phi_t + [g^{-1}g_t\ , \Phi] \\ -\Phi'_t &= \Phi_x + [g^{-1}g_x\ , \Phi] \end{aligned} \tag{43}$$

The integrability condition $(\Phi'_x)_t = (\Phi'_t)_x$, together with the equation $F[g]=0$, require that Φ be a solution of (42): $S(\Phi\,; g) = 0$. Similarly, by the integrability condition $(\Phi_t)_x = (\Phi_x)_t$ one finds, after a lengthy calculation: $S(\Phi'; g) = 0$.

In conclusion, for any g satisfying the PDE (40), the system (43) is a BT relating solutions Φ and Φ' of the symmetry condition (42) of this PDE; that is, relating different symmetries of the chiral field equation (40). Thus, if a symmetry characteristic $Q=g\Phi$ of (40) is known, a new characteristic $Q'=g\Phi'$ may be found by integrating the BT (43); the converse is also true. Since the BT (43) produces new symmetries from old ones, it may be regarded as a *recursion operator* for the PDE (40).

As an example, for any constant matrix M the choice $\Phi=M$ clearly satisfies the symmetry condition (42). This corresponds to the symmetry characteristic $Q=gM$. By integrating the BT (43) for Φ', we get $\Phi'=[X, M]$ and $Q'=g[X, M]$, where X is the "potential" of the PDE (40), defined by the system of PDEs

$$X_x = g^{-1}g_t \ , \quad -X_t = g^{-1}g_x \tag{44}$$

Note the *nonlocal* character of the BT-produced symmetry Q', due to the presence of the potential X. Indeed, as seen from (44), in order to find X one has to *integrate* the chiral field g with respect to the independent variables x and t. The above process can be continued indefinitely by repeated application of the recursion operator (43), leading to an infinite sequence of increasingly nonlocal symmetries.

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7. SUMMARY

Classically, Bäcklund transformations (BTs) have been developed as a useful tool for finding solutions of nonlinear PDEs, given that these equations are usually hard to solve by direct methods. By means of examples we saw that, starting with even the most trivial solution of a PDE, one may produce a highly nontrivial solution of this (or another) PDE by integrating the BT, without solving the original, nonlinear PDE directly (which, in most cases, is a much harder task).

A different use of BTs, that was recently proposed [7,8], concerns predominantly the solution of linear systems of PDEs. This method relies on the existence of parameter-dependent solutions of the linear PDEs expressing the integrability conditions of the BT. This time it is the BT itself (rather than its associated integrability conditions) whose solutions are sought.

An appropriate example for demonstrating this approach to the concept of a BT is furnished by the Maxwell equations of electromagnetism. We showed that this system of PDEs can be treated as a BT whose integrability conditions are the wave equations for the electric and the magnetic field. These wave equations have known, parameter-dependent solutions – monochromatic plane waves – with arbitrary amplitudes, frequencies and wave vectors playing the roles of the "parameters". By substituting these solutions into the BT, one may determine the required relations among the parameters in order that these plane waves also represent electromagnetic fields (i.e., in order that they be solutions of the Maxwell system). The results arrived at by this method are, of course, well known in advanced electrodynamics. The process of deriving them, however, is seen here in a new light by employing the concept of a BT.

BTs have also proven useful as *recursion operators* for deriving infinite sets of nonlocal symmetries and conservation laws of PDEs [2-6] (see also [11] and the references therein). Specifically, the BT produces an increasingly nonlocal sequence of symmetry characteristics, i.e., solutions of the linear equation expressing the symmetry condition (or "linearization") of a given PDE.

An interesting conclusion is that the concept of a BT, which has been proven useful for integrating nonlinear PDEs, may also have important applications in linear problems. Research on these matters is in progress.

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The Maxwell equations as a Bäcklund transformation

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Abstract

Bäcklund transformations (BTs) are a useful tool for integrating nonlinear partial differential equations (PDEs). However, the significance of BTs in linear problems should not be ignored. In fact, an important linear system of PDEs in Physics, namely, the Maxwell equations of electromagnetism, may be viewed as a BT relating the wave equations for the electric and the magnetic field, these equations representing integrability conditions for solution of the Maxwell system. We examine the BT property of this system in detail, both for the vacuum case and for the case of a linear conducting medium.

1. Introduction

Bäcklund transformations (BTs) are an effective tool for integrating partial differential equations (PDEs). They are particularly useful for obtaining solutions of nonlinear PDEs, given that these equations are often notoriously hard to solve by direct methods (see [1] and the references therein).

Generally speaking, given two PDEs – say (*a*) and (*b*) – for the unknown functions *u* and *v*, respectively, a BT relating these PDEs is a system of auxiliary PDEs containing both *u* and *v*, such that the consistency (*integrability*) of this system requires that the original PDEs (*a*) and (*b*) be separately satisfied. Then, if a solution of PDE (*a*) is known, a solution of PDE (*b*) is found simply by integrating the BT, without having to integrate the PDE (*b*) directly (which, presumably, is a much harder task).

In addition to being a solution-generating mechanism, BTs may also serve as *recursion operators* for obtaining infinite hierarchies of (generally nonlocal) symmetries and conservation laws of a PDE [1–7]. It is by this method that the full symmetry Lie algebra of the self-dual Yang-Mills equation was found [3,6].

In this article, the nature of which is mostly pedagogical, we adopt a somewhat different (in a sense, inverse) view of a BT, suitable for the treatment of linear problems. Suppose we are given a system of PDEs for the unknown functions *u* and *v*. Suppose, further, that the consistency of this system requires that two PDEs, one for *u* and one for *v*, be separately satisfied (thus, the given system is a BT connecting these PDEs). The PDEs are assumed to possess known solutions for *u* and *v*, each solution depending on a number of parameters. If, by a proper choice of the parameters, these functions are made to satisfy the original differential system, then a solution to this system has been found. In other words, we are seeking solutions of the given system by using known, parameter-dependent solutions of the individual PDEs expressing the integrability conditions of this system. Pairs of functions (*u*,*v*) satisfying the system will be said to represent *BT-conjugate solutions*.

This modified view of the concept of a BT has an important application in electromagnetism that serves as a paradigm for the significance of BTs in linear problems. As discussed in this paper, the Maxwell equations for a linear medium exactly fit this BT scheme. Indeed, as is well known, the consistency of the Maxwell system requires that the electric and the magnetic field satisfy separate wave equations. These equations have known, parameter-dependent solutions, namely, monochromatic plane waves with arbitrary amplitudes, wave vectors, frequencies, etc. (the "parameters" of the problem). By inserting these solutions into the Maxwell system, one may find the necessary conditions on the parameters in order that the plane waves for the two fields represent BT-conjugate solutions of Maxwell's equations.

The paper is organized as follows:

Section 2 reviews the classical concept of a BT. The solution-generating process by using a BT is demonstrated in a number of examples.

In Sec. 3 the concept of parametric, BT-conjugate solutions is introduced. A simple example illustrates the idea.

In Sec. 4 the Maxwell equations in empty space are shown to constitute a BT in the sense described in Sec. 3. For completeness of presentation (and for the benefit of the student) the process of constructing BT-conjugate plane-wave solutions is presented in detail.

Finally, in Sec. 5 the Maxwell system for a linear conducting medium is similarly examined.

The results of Secs. 4 and 5 are, of course, well known from classical electromagnetic theory. It is mathematically interesting, however, to revisit the problem of constructing solutions of Maxwell's equations from a novel point of view by using the concept of a BT and by treating the electric and the magnetic component of a plane e/m wave as BT-conjugate solutions.

2. Bäcklund transformations: definition and examples

The general idea of a Bäcklund transformation (BT) was explained in [1] (see also the references therein). Let us review the main points:

We consider two PDEs $P[u]=0$ and $Q[v]=0$, where the expressions $P[u]$ and $Q[v]$ may contain the unknown functions u and v, respectively, as well as some of their partial derivatives with respect to the independent variables. For simplicity, we assume that u and v are functions of only two variables x, t. Partial derivatives with respect to these variables will be denoted by using subscripts, e.g., u_x , u_t , u_{xx} , u_{tt} , u_{xt} , etc.

We also consider a system of coupled PDEs for u and v,

$$B_i\,[u,v]=0\ ,\quad i=1,2 \tag{1}$$

where the expressions $B_i\,[u,v]$ may contain u, v and certain of their partial derivatives with respect to x and t. The system (1) is assumed to be integrable for v (the two equations are compatible with each other for solution for v) when u satisfies the PDE $P[u]=0$. The solution v, then, satisfies the PDE $Q[v]=0$. Conversely, the system (1) is integrable for u if v satisfies the PDE $Q[v]=0$, the solution u then satisfying $P[u]=0$.

If the above assumptions are valid, we say that the system (1) constitutes a BT connecting solutions of $P[u]=0$ with solutions of $Q[v]=0$. In the special case where $P\equiv Q$, i.e., when u and v satisfy the same PDE, the system (1) is called an *auto-Bäcklund* transformation (auto-BT).

Suppose now that we seek solutions of the PDE $P[u]=0$. Also, assume that we possess a BT connecting solutions u of this equation with solutions v of the PDE $Q[v]=0$ (if $P\equiv Q$ the auto-BT connects solutions u and v of the same PDE). Let $v=v_0(x,t)$ be a known solution of $Q[v]=0$. The BT is then a system of equations for the unknown u:

$$B_i\,[u,v_0]=0\ ,\quad i=1,2 \tag{2}$$

Given that $Q[v_0]=0$, the system (2) is integrable for u and its solution satisfies the PDE $P[u]=0$. We may thus find a solution $u(x,t)$ of $P[u]=0$ without solving the equation itself, simply by integrating the BT (2) with respect to u. Of course, the use of this method is meaningful provided that we know a solution $v_0(x,t)$ of $Q[v]=0$ beforehand, as well as that integrating the system (2) for u is simpler than integrating the PDE $P[u]=0$ directly. If the transformation (2) is an auto-BT, then, starting with a known solution $v_0(x,t)$ of $P[u]=0$ and integrating the system (2), we find another solution $u(x,t)$ of the same equation.

Let us see some examples of using a BT to generate solutions of a PDE:

1. The *Cauchy-Riemann relations* of complex analysis,

$$u_x=v_y\quad (a)\qquad u_y=-v_x\quad (b) \tag{3}$$

(here, the variable t has been renamed y) constitute an auto-BT for the (linear) *Laplace equation*,

$$P\,[w]\equiv w_{xx}+w_{yy}=0 \tag{4}$$

Indeed, differentiating (3a) with respect to y and (3b) with respect to x, and demanding that the *integrability condition* $(u_x)_y=(u_y)_x$ be satisfied, we eliminate the variable u to find the consistency condition that must be obeyed by $v(x,y)$ in order that the system (3) be integrable for u:

$$P\,[v]\equiv v_{xx}+v_{yy}=0\ .$$

Conversely, eliminating v from the system (3) by using the integrability condition $(v_x)_y=(v_y)_x$, we find the necessary condition for u in order for the system to be integrable for v:

$$P\,[u]\equiv u_{xx}+u_{yy}=0\ .$$

Now, let $v_0(x,y)$ be a known solution of the Laplace equation (4). Substituting $v=v_0$ in the system (3), we can integrate the latter with respect to u to find another solution of the Laplace equation. For example, by choosing $v_0(x,y)=xy$ we find the solution $u(x,y)=(x^2-y^2)/2+C$.

2. The *Liouville equation* is written

$$P\,[u]\equiv u_{xt}-e^u=0\quad\Leftrightarrow\quad u_{xt}=e^u \tag{5}$$

Solving the PDE (5) directly is a difficult task in view of this equation's nonlinearity. A solution can be found, however, by using a BT. We thus consider an auxiliary function $v(x,t)$ and an associated linear PDE,

$$Q\,[v]\equiv v_{xt}=0 \tag{6}$$

We also consider the system of first-order PDEs,

$$\begin{aligned} u_x+v_x&=\sqrt{2}\ e^{(u-v)/2}\\ u_t-v_t&=\sqrt{2}\ e^{(u+v)/2}\end{aligned} \tag{7}$$

It can be shown that the self-consistency of the system (7) requires that u and v independently satisfy the PDEs (5) and (6), respectively. Thus, this system constitutes a BT connecting solutions of (5) and (6). Starting with the trivial solution $v=0$ of (6) and integrating the system

$$u_x=\sqrt{2}\ e^{u/2}\ ,\qquad u_t=\sqrt{2}\ e^{u/2}\ ,$$

we find a solution of (5):

$$u\,(x,t)=-2\ln\left(C-\frac{x+t}{\sqrt{2}}\right)\ .$$

3. The *"sine-Gordon" equation* has applications in various areas of Physics, such as in the study of crystalline solids, in the transmission of elastic waves, in magnetism, in elementary-particle models, etc. The equation (whose name

is a pun on the related linear Klein-Gordon equation) is written

$$u_{xt} = \sin u \tag{8}$$

As can be proven, the differential system

$$\begin{aligned} \frac{1}{2}(u+v)_x &= a \sin\left(\frac{u-v}{2}\right) \\ \frac{1}{2}(u-v)_t &= \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \end{aligned} \tag{9}$$

[where a ($\neq 0$) is an arbitrary real constant] is a parametric auto-BT for the PDE (8). Starting with the trivial solution $v=0$ of $v_{xt}=\sin v$, and integrating the system

$$u_x = 2a \sin\frac{u}{2} \quad , \quad u_t = \frac{2}{a} \sin\frac{u}{2} \quad ,$$

we obtain a new solution of (8):

$$u(x,t) = 4 \arctan\left\{ C \exp\left(a x + \frac{t}{a} \right) \right\} \; .$$

3. BT-conjugate solutions

Consider a system of coupled PDEs for the functions u and v of two independent variables x, y:

$$B_i [u,v] = 0 \; , \quad i = 1, 2 \tag{10}$$

Assume that the integrability of this system for both u and v requires that the following PDEs be independently satisfied:

$$P[u] = 0 \quad (a) \qquad Q[v] = 0 \quad (b) \tag{11}$$

That is, the system (10) represents a BT connecting the PDEs (11). Assume, further, that the PDEs (11) possess parameter-dependent solutions of the form

$$\begin{aligned} u &= f(x, y; \alpha, \beta, \gamma, \ldots) \; , \\ v &= g(x, y; \kappa, \lambda, \mu, \ldots) \end{aligned} \tag{12}$$

where α, β, κ, λ, etc., are (real or complex) parameters. If values of these parameters can be determined for which u and v satisfy the system (10), we say that the solutions u and v of the PDEs (11a) and (11b), respectively, are *conjugate through the BT* (10) (or *BT-conjugate*, for short).

Let us see an example: Going back to the Cauchy-Riemann relations (3), we try the following parametric solutions of the Laplace equation (4):

$$\begin{aligned} u(x,y) &= \alpha (x^2 - y^2) + \beta x + \gamma y \; , \\ v(x,y) &= \kappa xy + \lambda x + \mu y \; . \end{aligned}$$

Substituting these into the BT (3), we find that $\kappa=2\alpha$, $\mu=\beta$ and $\lambda=-\gamma$. Therefore, the solutions

$$\begin{aligned} u(x,y) &= \alpha (x^2 - y^2) + \beta x + \gamma y \; , \\ v(x,y) &= 2\alpha xy - \gamma x + \beta y \end{aligned}$$

of the Laplace equation are BT-conjugate through the Cauchy-Riemann relations.

As a counter-example, let us try a different combination:

$$u(x,y) = \alpha xy \quad , \quad v(x,y) = \beta xy \; .$$

Inserting these into the system (3) and taking into account the independence of x and y, we find that the only possible values of the parameters α and β are $\alpha=\beta=0$, so that $u(x,y)=v(x,y)=0$. Thus, no non-trivial BT-conjugate solutions exist in this case.

4. Application to the Maxwell equations in empty space

As is well known, according to the Maxwell theory all electromagnetic (e/m) disturbances propagate in space as waves running at the speed of light. It is interesting from the mathematical point of view that the vacuum wave equations for the electric and the magnetic field are connected to each other through the Maxwell system of equations in much the same way two PDEs are connected via a Bäcklund transformation. In fact, certain parameter-dependent solutions of the two wave equations are BT-conjugate through the Maxwell system.

In empty space, where no charges or currents (whether free or bound) exist, the Maxwell equations are written in S.I. units [8]:

$$\begin{aligned} &(a) \;\; \vec{\nabla}\cdot\vec{E} = 0 \qquad (c) \;\; \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &(b) \;\; \vec{\nabla}\cdot\vec{B} = 0 \qquad (d) \;\; \vec{\nabla}\times\vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{13}$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. In order that this system of PDEs be self-consistent (thus integrable for the two fields), certain consistency conditions (or *integrability conditions*) must be satisfied. Four are satisfied automatically:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0 \quad , \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \quad ,$$

$$(\vec{\nabla}\cdot\vec{E})_t = \vec{\nabla}\cdot\vec{E}_t \quad , \quad (\vec{\nabla}\cdot\vec{B})_t = \vec{\nabla}\cdot\vec{B}_t \quad .$$

Two others read:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \tag{14}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \tag{15}$$

Taking the *rot* of (13*c*) and using (14), (13*a*) and (13*d*), we find:

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{16}$$

Similarly, taking the *rot* of (13*d*) and using (15), (13*b*) and (13*c*), we get:

$$\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{17}$$

No new information is furnished by the remaining two integrability conditions,

$$(\vec{\nabla}\times\vec{E})_t=\vec{\nabla}\times\vec{E}_t \quad , \quad (\vec{\nabla}\times\vec{B})_t=\vec{\nabla}\times\vec{B}_t \quad .$$

Putting

$$\varepsilon_0\mu_0\equiv\frac{1}{c^2} \quad \Leftrightarrow \quad c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \tag{18}$$

we rewrite Eqs. (16) and (17) in wave-equation form:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{19}$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{20}$$

The PDEs (19) and (20) are consistency conditions that must be separately satisfied by $\vec{E}$ and $\vec{B}$ in order that the differential system (13) be integrable for either field, given the value of the other field. In other words, the system (13) is a BT relating solutions of the wave equations (19) and (20).

It should be noted carefully that the BT (13) is *not* an *auto*-BT! Indeed, although the PDEs (19) and (20) look similar, they concern *different* fields with different physical dimensions and physical properties. A true auto-BT should connect similar objects (such as, e.g., different mathematical expressions for the electric field).

The above wave equations admit plane-wave solutions of the form $\vec{F}(\vec{k}\cdot\vec{r}-\omega t)$, with

$$\frac{\omega}{k}=c \quad \text{where} \quad k=|\vec{k}| \tag{21}$$

The simplest such solutions are *monochromatic plane waves* of angular frequency *ω*, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned}&\vec{E}(\vec{r},t)=\vec{E}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (a)\\ &\vec{B}(\vec{r},t)=\vec{B}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \qquad (b)\end{aligned} \tag{22}$$

where the $\vec{E}_0$ and $\vec{B}_0$ represent constant complex amplitudes. Since all constants appearing in equations (22) (that is, amplitudes, frequency and wave vector) can be arbitrarily chosen, they can be regarded as *parameters* on which the solutions (22) of the wave equations depend.

Clearly, although every pair of fields $(\vec{E},\vec{B})$ that satisfies the Maxwell equations (13) also satisfies the respective wave equations (19) and (20), the converse is not true. This means that the solutions (22) of the wave equation are not *a priori* solutions of the Maxwell system of equations (i.e., do not represent e/m fields). This problem can be remedied, however, by appropriate choice of the parameters. To this end, we substitute the general solutions (22) into the system (13) in order to find the extra conditions this system requires; that is, in order to make the two functions in (22) BT-conjugate solutions of the respective wave equations (19) and (20).

Substituting (22*a*) and (22*b*) into (13*a*) and (13*b*), respectively, and taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, we have:

$$(\vec{E}_0e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;\Rightarrow\; (\vec{k}\cdot\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\ ,$$
$$(\vec{B}_0e^{-i\omega t})\cdot\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=0 \;\Rightarrow\; (\vec{k}\cdot\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0\ ,$$

so that

$$\vec{k}\cdot\vec{E}_0=0\ , \quad \vec{k}\cdot\vec{B}_0=0\ . \tag{23}$$

Physically, this means that the monochromatic plane e/m wave is a *transverse* wave.

Next, substituting (22*a*) and (22*b*) into (13*c*) and (13*d*), we find:

$$e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{E}_0=i\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{E}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\omega\,\vec{B}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ ,$$

$$e^{-i\omega t}(\vec{\nabla}e^{i\vec{k}\cdot\vec{r}})\times\vec{B}_0=-i\omega\,\varepsilon_0\mu_0\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$
$$(\vec{k}\times\vec{B}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}=-\frac{\omega}{c^2}\vec{E}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ ,$$

so that

$$\vec{k}\times\vec{E}_0=\omega\,\vec{B}_0\ ,\qquad \vec{k}\times\vec{B}_0=-\frac{\omega}{c^2}\,\vec{E}_0 \tag{24}$$

This means that the fields $\vec{E}$ and $\vec{B}$ are normal to each other as well as being normal to the direction of propagation. It can be seen that the two vector equations in (24) are not independent of each other; indeed, cross-multiplying the first relation by $\vec{k}$ we get the second one.

Introducing a unit vector $\hat{\tau}$ in the direction of the wave vector $\vec{k}$,

$$\hat{\tau}=\vec{k}/k\qquad (k=|\vec{k}|=\omega/c)\ ,$$

we rewrite the first of Eqs. (24) as

$$\vec{B}_0=\frac{k}{\omega}(\hat{\tau}\times\vec{E}_0)=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\ .$$

The BT-conjugate solutions in (22) are now written:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}\ ,\\ \vec{B}(\vec{r},t)&=\frac{1}{c}(\hat{\tau}\times\vec{E}_0)\exp\{\,i(\vec{k}\cdot\vec{r}-\omega t)\}\\ &=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{25}$$

As constructed, the complex vector fields in (25) satisfy the Maxwell system (13), which is a homogeneous linear system with real coefficients. Evidently, the real parts of these fields also satisfy this system. To find the expressions for the real solutions (which, after all, carry the physics of the situation) we take the simplest case of a linearly polarized e/m wave and write:

$$\vec{E}_0=\vec{E}_{0,R}\,e^{\,i\alpha} \tag{26}$$

where the vector $\vec{E}_{0,R}$ and the number α are real. The *real* versions of the fields (25), then, read:

$$\begin{aligned}\vec{E}&=\vec{E}_{0,R}\cos(\vec{k}\cdot\vec{r}-\omega\,t+\alpha)\ ,\\ \vec{B}&=\frac{1}{c}(\hat{\tau}\times\vec{E}_{0,R})\cos(\vec{k}\cdot\vec{r}-\omega\,t+\alpha)\\ &=\frac{1}{c}\,\hat{\tau}\times\vec{E}\end{aligned} \tag{27}$$

We note, in particular, that the fields $\vec{E}$ and $\vec{B}$ "oscillate" in phase.

Our results for the Maxwell equations in vacuum can be extended to the case of a *linear non-conducting medium* upon replacement of ε_0 and μ_0 with ε and μ, respectively. The speed of propagation of the e/m wave is, in this case,

$$\upsilon=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon\mu}}\ .$$

5. The Maxwell system for a linear conducting medium

In a linear conducting medium of conductivity σ, in which Ohm's law is satisfied, $\vec{J}_f=\sigma\vec{E}$ (where $\vec{J}_f$ is the free current density), the Maxwell equations read [8]:

$$\begin{aligned}&(a)\ \ \vec{\nabla}\cdot\vec{E}=0\qquad (c)\ \ \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ &(b)\ \ \vec{\nabla}\cdot\vec{B}=0\qquad (d)\ \ \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\,\frac{\partial\vec{E}}{\partial t}\end{aligned} \tag{28}$$

By the integrability conditions

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\vec{E})&=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}\ ,\\ \vec{\nabla}\times(\vec{\nabla}\times\vec{B})&=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}\ ,\end{aligned}$$

we get the *modified wave equations*

$$\begin{aligned}&\nabla^2\vec{E}-\varepsilon\mu\,\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\,\frac{\partial\vec{E}}{\partial t}=0\\ &\nabla^2\vec{B}-\varepsilon\mu\,\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\,\frac{\partial\vec{B}}{\partial t}=0\end{aligned} \tag{29}$$

No new information is furnished by the remaining integrability conditions (cf. Sec. 4).

We observe that the linear differential system (28) is a BT relating solutions of the wave equations (29) (as explained in the previous section, this BT is *not* an auto-BT). As in the vacuum case, we seek BT-conjugate such solutions. As can be verified by direct substitution into Eqs. (29), these PDEs admit parametric plane-wave solutions of the form

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega t)\}\\ &=\vec{E}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)\ ,\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{-s\,\hat{\tau}\cdot\vec{r}+i\,(\vec{k}\cdot\vec{r}-\omega\,t)\}\\ &=\vec{B}_0\exp\left\{\left(i-\frac{s}{k}\right)\vec{k}\cdot\vec{r}\right\}\exp(-i\omega t)\end{aligned} \tag{30}$$

where $\hat{\tau}$ is the unit vector in the direction of the wave vector $\vec{k}$,

$$\hat{\tau} = \vec{k} / k \quad (k = |\vec{k}| = \omega / \upsilon)$$

(υ is the speed of propagation of the wave inside the conducting medium) and where, for given physical characteristics ε, μ, σ of the medium, the parameters s, k and ω satisfy the algebraic system

$$\begin{aligned} & s^2 - k^2 + \varepsilon \mu \omega^2 = 0 \ , \\ & \mu \sigma \omega - 2sk = 0 \end{aligned} \qquad (31)$$

Up to this point the complex amplitudes $\vec{E}_0$ and $\vec{B}_0$ in relations (30) are arbitrary and the vector fields (30) are not *a priori* solutions of the Maxwell equations (28), thus are not yet BT-conjugate solutions of the respective wave equations in (29). To find the restrictions these amplitudes must satisfy, we insert Eqs. (30) into the system (28). With the aid of the relation

$$\vec{\nabla} e^{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}} = \left(i - \frac{s}{k}\right) \vec{k} \, e^{\left(i - \frac{s}{k}\right) \vec{k} \cdot \vec{r}} \ ,$$

it is not hard to show that (28*a*) and (28*b*) impose the conditions

$$\vec{k} \cdot \vec{E}_0 = 0 \ , \quad \vec{k} \cdot \vec{B}_0 = 0 \qquad (32)$$

Again, this means that the e/m wave is a transverse wave.

Substituting (30) into (28*c*) and (28*d*), we find two more conditions:

$$(k + is) \, \hat{\tau} \times \vec{E}_0 = \omega \vec{B}_0 \qquad (33)$$

$$(k + is) \, \hat{\tau} \times \vec{B}_0 = -(\varepsilon \mu \omega + i \mu \sigma) \vec{E}_0 \qquad (34)$$

However, (34) is not an independent equation since it can be reproduced by cross-multiplication of (33) by $\hat{\tau}$ and use of relations (31).

The BT-conjugate solutions of the wave equations (29) are now written:

$$\begin{aligned} & \vec{E}(\vec{r}, t) = \vec{E}_0 \, e^{-s \hat{\tau} \cdot \vec{r}} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \ , \\ & \vec{B}(\vec{r}, t) = \frac{k + is}{\omega} \, (\hat{\tau} \times \vec{E}_0) \, e^{-s \hat{\tau} \cdot \vec{r}} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \qquad (35)$$

To find the corresponding real solutions, we assume linear polarization of the e/m wave and set, as before,

$$\vec{E}_0 = \vec{E}_{0,R} \, e^{i\alpha} \ .$$

We also set

$$\begin{aligned} & k + i\,s = |k + i\,s| e^{i\varphi} = \sqrt{k^2 + s^2} \; e^{i\varphi} \ ; \\ & \tan \varphi = s / k \ . \end{aligned}$$

Taking the real parts of Eqs. (35), we finally have:

$$\begin{aligned} & \vec{E}(\vec{r}, t) = \vec{E}_{0,R} \, e^{-s \hat{\tau} \cdot \vec{r}} \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha) \ , \\ & \vec{B}(\vec{r}, t) = \frac{\sqrt{k^2 + s^2}}{\omega} \, (\hat{\tau} \times \vec{E}_{0,R}) \, e^{-s \hat{\tau} \cdot \vec{r}} \cos(\vec{k} \cdot \vec{r} - \omega t + \alpha + \varphi) \ . \end{aligned}$$

6. Summary and concluding remarks

Bäcklund transformations (BTs) were originally devised as a tool for finding solutions of nonlinear partial differential equations (PDEs). They were later also proven useful as nonlocal recursion operators for constructing infinite sequences of symmetries and conservation laws of certain PDEs [2–7].

Generally speaking, a BT is a system of PDEs connecting two fields that are required to independently satisfy two respective PDEs in order for the system to be integrable for either field. If a solution of either PDE is known, then a solution of the other PDE is obtained by integrating the BT, without having to actually solve the latter PDE explicitly (which, presumably, would be a much harder task). In the case where the two PDEs are identical, an auto-BT produces new solutions of a PDE from old ones.

As described above, a BT is an auxiliary tool for finding solutions of a given (usually nonlinear) PDE, using known solutions of the same or another PDE. In this article, however, we approached the BT concept differently by actually inverting the problem. According to this scheme, it is the solutions of the BT itself that we are after, having parameter-dependent solutions of the PDEs that express the integrability conditions at hand. By a proper choice of the parameters, a pair of solutions of these PDEs may possibly be found that satisfies the given BT. These solutions are then said to be *conjugate* with respect to the BT.

A pedagogical paradigm for demonstrating this particular approach to the concept of a BT is offered by the Maxwell system of equations of electromagnetism. We showed that this system can be thought of as a BT whose integrability conditions are the wave equations for the electric and the magnetic field. These wave equations have known, parameter-dependent solutions (monochromatic plane waves) with arbitrary amplitudes, frequencies, wave vectors, etc. By substituting these solutions into the BT, one may determine the required relations among the parameters in order that the plane waves also represent electromagnetic fields, i.e., are BT-conjugate solutions of the Maxwell system. The results arrived at by this method are, of course, well known in advanced electrodynamics. The process of deriving them, however, is seen here in a new light by employing the concept of a BT.

We remark that the physical situation was examined from the point of view of a fixed inertial observer. Thus, since no spacetime transformations were involved, we used the classical form of the Maxwell equations (with $\vec{E}$ and $\vec{B}$ retaining their individual characters) rather than the manifestly covariant form of these equations.

An interesting conclusion is that the concept of a Bäcklund transformation, which has been proven extremely useful for finding solutions of nonlinear PDEs, can in certain cases also prove useful for integrating *linear systems* of PDEs. Such systems appear often in Physics and Electrical Engineering (see, e.g., [9]) and it would certainly be of interest to explore the possibility of using BT methods for their integration.

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Plane-wave solutions of Maxwell's equations: An educational note

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Synopsis

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

Plane-wave solutions of Maxwell's equations: An educational note

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Abstract

In electrodynamics courses and textbooks, the properties of plane electromagnetic waves in both conducting and non-conducting media are typically studied from the point of view of the prototype case of a monochromatic plane wave. In this note an approach is suggested that starts from more general considerations and better exploits the independence of the Maxwell equations.

1. Introduction

Plane electromagnetic (e/m) waves constitute a significant type of solution of the time-dependent Maxwell equations. A standard educational approach in courses and textbooks (at both the intermediate [1-4] and the advanced [5,6] level; see also [7,8]) is to examine the prototype case of a monochromatic plane wave in both a conducting and a non-conducting medium.

In this note a more general approach to the problem is described that makes minimal initial assumptions regarding the specific functional forms of the plane waves representing the electric and the magnetic field. The only assumption one does need to make from the outset is that both fields (electric and magnetic) are expressible in integral form as linear superpositions of monochromatic waves. In particular, it is not even necessary to *a priori* require that the plane waves representing the two fields travel in the same direction.

In Section 2 we review the case of a monochromatic plane e/m wave in empty space. A more general (non-monochromatic) treatment of the plane-wave propagation problem in empty space is then described in Sec. 3. In Sec. 4 this general approach is extended to plane-wave solutions in the case of a conducting medium; an interesting difference from the monochromatic case is noted.

2. The monochromatic-wave description for empty space

In empty space, where no charges or currents (whether free or bound) exist, the Maxwell equations are written (in S.I. units)

$$(a)\quad \vec{\nabla}\cdot\vec{E}=0 \qquad (c)\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$(b)\quad \vec{\nabla}\cdot\vec{B}=0 \qquad (d)\quad \vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \qquad (1)$$

where $\vec{E}$ and $\vec{B}$ are the electric and the magnetic field, respectively. By applying the identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \ ,$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \, ,$$

we obtain separate wave equations for $\vec{E}$ and $\vec{B}$:

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0 \qquad (2)$$

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \qquad (3)$$

where

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad (4)$$

We try monochromatic plane-wave solutions of (2) and (3), of angular frequency ω, propagating in the direction of the wave vector $\vec{k}$:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\vec{E}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \quad (a)\\ \vec{B}(\vec{r},t)&=\vec{B}_0\exp\{i(\vec{k}\cdot\vec{r}-\omega t)\} \quad (b)\end{aligned} \qquad (5)$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant complex amplitudes, and where

$$\frac{\omega}{k}=c \qquad (k=|\vec{k}\,|) \qquad (6)$$

The general solutions (5) do not *a priori* represent an e/m field. To find the extra constraints required, we must substitute Eqs. (5) into the Maxwell system (1). By taking into account that $\vec{\nabla}e^{i\vec{k}\cdot\vec{r}}=i\vec{k}\,e^{i\vec{k}\cdot\vec{r}}$, the *div* equations (1*a*) and (1*b*) yield

$$\vec{k}\cdot\vec{E}=0 \quad (a) \qquad \vec{k}\cdot\vec{B}=0 \quad (b) \qquad (7)$$

while the *rot* equations (1*c*) and (1*d*) give

$$\vec{k}\times\vec{E}=\omega\vec{B} \quad (a) \qquad \vec{k}\times\vec{B}=-\frac{\omega}{c^2}\vec{E} \quad (b) \qquad (8)$$

Now, we notice that the four equations (7)–(8) do not form an independent set since (7*b*) and (8*b*) can be reproduced by using (7*a*) and (8*a*). Indeed, taking the dot product of (8*a*) with $\vec{k}$ we get (7*b*), while taking the cross product of (8*a*) with $\vec{k}$, and using (7*a*) and (6), we find (8*b*).

So, from 4 independent Maxwell equations we obtained only 2 independent pieces of information. This happened because we “fed” our trial solutions (5) with more information than necessary, in anticipation of results that follow *a posteriori* from Maxwell’s equations. Thus, we assumed from the outset that the two waves (electric and magnetic) have similar simple functional forms and propagate in the same direction. By relaxing these initial assumptions, our analysis acquires a richer and much more interesting structure.

3. A more general approach for empty space

Let us assume, more generally, that the fields $\vec{E}$ and $\vec{B}$ represent plane waves propagating in empty space in the directions of the unit vectors $\hat{\tau}$ and $\hat{\sigma}$, respectively:

$$\vec{E}(\vec{r},t) = \vec{F}(\hat{\tau}\cdot\vec{r} - ct) \ , \quad \vec{B}(\vec{r},t) = \vec{G}(\hat{\sigma}\cdot\vec{r} - ct) \tag{9}$$

Furthermore, assume that the functions $\vec{F}$ and $\vec{G}$ can be expressed as linear combinations of monochromatic plane waves of the form (5), for continuously varying values of *k* and *ω*, where *ω=ck*, according to (6). Then $\vec{E}$ and $\vec{B}$ can be written in Fourier-integral form, as follows:

$$\begin{aligned}\vec{E} &= \int \vec{E}_0(k)\, e^{ik(\hat{\tau}\cdot\vec{r} - ct)} dk \\ \vec{B} &= \int \vec{B}_0(k)\, e^{ik(\hat{\sigma}\cdot\vec{r} - ct)} dk\end{aligned} \tag{10}$$

In general, the integration variable *k* is assumed to run from 0 to +∞. For notational economy, the limits of integration with respect to *k* will not be displayed explicitly.

By setting

$$u = \hat{\tau}\cdot\vec{r} - ct \ , \quad v = \hat{\sigma}\cdot\vec{r} - ct \tag{11}$$

we write

$$\begin{aligned}\vec{E}(u) &= \int \vec{E}_0(k)\, e^{iku} dk \\ \vec{B}(v) &= \int \vec{B}_0(k)\, e^{ikv} dk\end{aligned} \tag{12}$$

We note that

$$\vec{\nabla} e^{iku} = ik\hat{\tau} e^{iku} \ , \quad \vec{\nabla} e^{ikv} = ik\hat{\sigma} e^{ikv} \tag{13}$$

By using (12) and (13) we find that

$$\vec{\nabla}\cdot\vec{E} = \int ik\hat{\tau}\cdot\vec{E}_0(k)\, e^{iku} dk \ , \quad \vec{\nabla}\cdot\vec{B} = \int ik\hat{\sigma}\cdot\vec{B}_0(k)\, e^{ikv} dk \ ,$$

$$\vec{\nabla}\times\vec{E} = \int ik\hat{\tau}\times\vec{E}_0(k)\, e^{iku} dk \ , \quad \vec{\nabla}\times\vec{B} = \int ik\hat{\sigma}\times\vec{B}_0(k)\, e^{ikv} dk \ .$$

Moreover, we have that

$$\frac{\partial\vec{E}}{\partial t} = -\int i\omega\vec{E}_0(k)\, e^{iku} dk \ , \quad \frac{\partial\vec{B}}{\partial t} = -\int i\omega\vec{B}_0(k)\, e^{ikv} dk$$

where, as always, *ω=ck*.

The two Gauss' laws (1*a*) and (1*b*) yield

$$\int k\hat{\tau}\cdot\vec{E}_0(k)\,e^{iku}dk=0 \quad \text{and} \quad \int k\hat{\sigma}\cdot\vec{B}_0(k)\,e^{ikv}dk=0\,,$$

respectively. In order that these relations be valid identically for all *u* and all *v*, respectively, we must have

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\sigma}\cdot\vec{B}_0(k)=0\,, \ \text{ for all } k \tag{14}$$

From Faraday's law (1*c*) and the Ampère-Maxwell law (1*d*) we obtain two more integral equations:

$$\int k\hat{\tau}\times\vec{E}_0(k)\,e^{iku}dk=\int \omega\vec{B}_0(k)\,e^{ikv}dk \tag{15}$$

$$\int k\hat{\sigma}\times\vec{B}_0(k)\,e^{ikv}dk=-\int \frac{\omega}{c^2}\vec{E}_0(k)\,e^{iku}dk \tag{16}$$

where we have taken into account Eq. (4).

Taking the cross product of (15) with $\hat{\sigma}$ and using (16), we find the integral relation

$$\int k[(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0]\,e^{iku}dk=-\int k\vec{E}_0\,e^{iku}dk\,.$$

This is true for all *u* if

$$(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}-(\hat{\sigma}\cdot\hat{\tau})\vec{E}_0=-\vec{E}_0 \;\Rightarrow\; (\hat{\sigma}\cdot\hat{\tau}-1)\vec{E}_0=(\hat{\sigma}\cdot\vec{E}_0)\,\hat{\tau}\,.$$

Given that, by (14), $\vec{E}_0$ and $\hat{\tau}$ are mutually perpendicular, the above relation can only be valid if $\hat{\sigma}\cdot\hat{\tau}=1$ and $\hat{\sigma}\cdot\vec{E}_0=0$. This, in turn, can only be satisfied if $\hat{\sigma}=\hat{\tau}$. The same conclusion is reached by taking the cross product of (16) with $\hat{\tau}$ and by using (15) as well as the fact that $\vec{B}_0$ is normal to $\hat{\sigma}$. From (11) we then have that

$$u=v=\hat{\tau}\cdot\vec{r}-ct$$

so that relations (12) become

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int \vec{E}_0(k)\,e^{iku}dk=\int \vec{E}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\\ \vec{B}(\vec{r},t)&=\int \vec{B}_0(k)\,e^{iku}dk=\int \vec{B}_0(k)\,e^{ik(\hat{\tau}\cdot\vec{r}-ct)}dk\end{aligned} \tag{17}$$

Equations (14) are now rewritten as

$$\hat{\tau}\cdot\vec{E}_0(k)=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k)=0\,, \ \text{ for all } k \tag{18}$$

Furthermore, in order that (15) and (16) (with *u* and $\hat{\tau}$ in place of *v* and $\hat{\sigma}$, respectively) be identically valid for all *u*, we must have

$$k\,\hat{\tau}\times\vec{E}_0(k)=\omega\vec{B}_0(k) \;\Leftrightarrow\; \hat{\tau}\times\vec{E}_0(k)=c\vec{B}_0(k) \tag{19}$$

and

$$k\,\hat{\tau}\times\vec{B}_0(k)=-\frac{\omega}{c^2}\vec{E}_0(k) \;\Leftrightarrow\; \hat{\tau}\times\vec{B}_0(k)=-\frac{1}{c}\vec{E}_0(k) \tag{20}$$

for all k, where $k=\omega/c$. Notice, however, that (19) and (20) are not independent equations, since (20) is essentially the cross product of (19) with $\hat{\tau}$.

In summary, the general plane-wave solutions to the Maxwell system (1) are given by relations (17) with the additional constraints (18) and (19). This is, of course, a well-known result, derived here by starting with more general assumptions and by best exploiting the independence [9] of the Maxwell equations.

Let us summarize our main findings:

1. The fields $\vec{E}$ and $\vec{B}$ are plane waves traveling in the same direction, defined by the unit vector $\hat{\tau}$; these fields satisfy the Maxwell equations in empty space.

2. The e/m wave $(\vec{E},\vec{B})$ is a *transverse* wave. Indeed, from equations (17) and the orthogonality relations (18) it follows that

$$\hat{\tau}\cdot\vec{E}=0 \quad\text{and}\quad \hat{\tau}\cdot\vec{B}=0 \tag{21}$$

3. The fields $\vec{E}$ and $\vec{B}$ are mutually perpendicular. Moreover, the $(\vec{E},\vec{B},\hat{\tau})$ define a right-handed rectangular system. Indeed, by cross-multiplying (17) with $\hat{\tau}$ and by using (19) and (20), we find:

$$\hat{\tau}\times\vec{E}=c\vec{B}\;, \quad \hat{\tau}\times\vec{B}=-\frac{1}{c}\vec{E} \tag{22}$$

4. Taking *real values* of (21) and (22), we have:

$$\hat{\tau}\cdot\mathrm{Re}\vec{E}=0\;, \quad \hat{\tau}\cdot\mathrm{Re}\vec{B}=0 \quad\text{and}\quad \hat{\tau}\times\mathrm{Re}\vec{E}=c\,\mathrm{Re}\vec{B} \tag{23}$$

The magnitude of the last vector equation in (23) gives a relation between the instantaneous values of the electric and the magnetic field:

$$|\mathrm{Re}\vec{E}|=c\,|\mathrm{Re}\vec{B}| \tag{24}$$

The above results for empty space can be extended in a straightforward way to the case of a *linear, non-conducting, non-dispersive* medium upon replacement of ε_0 and μ_0 with ε and μ, respectively [3]. The (frequency-independent) speed of propagation of the plane e/m wave in this case is $\upsilon=1/(\varepsilon\mu)^{1/2}$.

4. The case of a conducting medium

The Maxwell equations for a conducting medium of conductivity σ may be written as follows [1,3]:

$$(a) \quad \vec{\nabla}\cdot\vec{E}=0 \qquad (c) \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$(b) \quad \vec{\nabla}\cdot\vec{B}=0 \qquad (d) \quad \vec{\nabla}\times\vec{B}=\mu\sigma\vec{E}+\varepsilon\mu\frac{\partial\vec{E}}{\partial t} \tag{25}$$

By using the vector identities

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} \ ,$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} \, ,$$

the relations (25) lead to the *modified wave equations*

$$\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0 \tag{26}$$

$$\nabla^2\vec{B}-\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}-\mu\sigma\frac{\partial\vec{B}}{\partial t}=0 \tag{27}$$

Guided by our monochromatic-wave approach to the problem in [7,8], we now try a more general, integral form of solution of the above wave equations:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{E}_0(k)\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,e^{-s\hat{\tau}\cdot\vec{r}}e^{i(k\hat{\tau}\cdot\vec{r}-\omega t)}dk=\int\vec{B}_0(k)\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\,dk\end{aligned} \tag{28}$$

where s is a real parameter related to the conductivity of the medium. As in the vacuum case, the unit vector $\hat{\tau}$ indicates the direction of propagation of the wave. Notice that we have assumed from the outset that both waves – electric and magnetic – propagate in the same direction, in view of the fact that our results must agree with those for a non-conducting medium (in particular, for the vacuum) upon setting s=0.

It is convenient to set

$$\exp\{(ik-s)\,\hat{\tau}\cdot\vec{r}-i\omega t\}\equiv A(\vec{r},t) \tag{29}$$

Then, Eq. (28) takes on the form

$$\begin{aligned}\vec{E}(\vec{r},t)&=\int\vec{E}_0(k)\,A(\vec{r},t)\,dk\\ \vec{B}(\vec{r},t)&=\int\vec{B}_0(k)\,A(\vec{r},t)\,dk\end{aligned} \tag{30}$$

The following relations can be easily proven:

$$\vec{\nabla}A(\vec{r},t)=(ik-s)\,\hat{\tau}A(\vec{r},t) \tag{31}$$

$$\nabla^2A(\vec{r},t)=(s^2-k^2-2isk)A(\vec{r},t) \tag{32}$$

Moreover,

$$\frac{\partial}{\partial t}A(\vec{r},t) = -i\omega A(\vec{r},t) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}A(\vec{r},t) = -\omega^2 A(\vec{r},t)\ .$$

From (26) we get

$$\int [(s^2 - k^2 + \varepsilon\mu\omega^2) + i(\mu\sigma\omega - 2sk)]\,\vec{E}_0(k)A(\vec{r},t)\,dk = 0$$

[a similar integral relation is found from (27)]. This will be identically satisfied for all $\vec{r}$ and t if

$$s^2 - k^2 + \varepsilon\mu\omega^2 = 0 \quad \text{and} \quad \mu\sigma\omega - 2sk = 0 \tag{33}$$

By using relations (33), ω and s can be expressed as functions of k, as required in order that the integral relations (28) make sense. Notice, in particular, that, by the second relation (33), s=0 if σ=0 (non-conducting medium). Then, by the first relation, $\omega/k=1/(\varepsilon\mu)^{1/2}$, which is the familiar expression for the speed of propagation of an e/m wave in a non-conducting medium [3].

From the two Gauss' laws (25*a*) and (25*b*) we get the corresponding integral relations

$$\int (ik - s)\,\hat{\tau}\cdot\vec{E}_0(k)A(\vec{r},t)\,dk = 0\ ,$$
$$\int (ik - s)\,\hat{\tau}\cdot\vec{B}_0(k)A(\vec{r},t)\,dk = 0\ .$$

These will be identically satisfied for all $\vec{r}$ and t if

$$\hat{\tau}\cdot\vec{E}_0(k) = 0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}_0(k) = 0, \quad \text{for all } k \tag{34}$$

From (25*c*) and (25*d*) we find

$$\int (ik - s)\,\hat{\tau}\times\vec{E}_0(k)A(\vec{r},t)\,dk = \int i\omega\vec{B}_0(k)A(\vec{r},t)\,dk$$

and

$$\int (ik - s)\,\hat{\tau}\times\vec{B}_0(k)A(\vec{r},t)\,dk = \int (\mu\sigma - i\varepsilon\mu\omega)\vec{E}_0(k)A(\vec{r},t)\,dk\ ,$$

respectively. To satisfy these for all $\vec{r}$ and t, we require that

$$(k + is)\,\hat{\tau}\times\vec{E}_0(k) = \omega\vec{B}_0(k) \tag{35}$$

and

$$(k + is)\,\hat{\tau}\times\vec{B}_0(k) = -(\varepsilon\mu\omega + i\mu\sigma)\vec{E}_0(k) \tag{36}$$

Note, however, that (36) is not an independent equation since it can be reproduced by cross-multiplying (35) with $\hat{\tau}$ and by taking into account Eqs. (33) and (34).

We note the following:

1. From (30) and (34) we have that

$$\hat{\tau}\cdot\vec{E}=0 \quad \text{and} \quad \hat{\tau}\cdot\vec{B}=0 \tag{37}$$

or, in real form, $\hat{\tau}\cdot \mathrm{Re}\,\vec{E}=0$ and $\hat{\tau}\cdot \mathrm{Re}\,\vec{B}=0$. This means that both $\mathrm{Re}\,\vec{E}$ and $\mathrm{Re}\,\vec{B}$ are normal to the direction of propagation of the wave.

2. From (30) and (35) we get

$$\hat{\tau}\times\vec{E}=\int \frac{\omega}{k+is}\vec{B}_0(k)A(\vec{r},t)\,dk \tag{38}$$

The integral on the right-hand side of (38) is, generally, not a vector parallel to $\vec{B}$. Now, in the limit of negligible conductivity (σ=0) the relations (33) give s=0 and $\omega/k=1/(\varepsilon\mu)^{1/2}$. The ratio ω/k represents the speed of propagation υ in the non-conducting medium, for the frequency ω. If the medium is *non-dispersive*, the speed $\upsilon=\omega/k$ is constant, independent of frequency. Then Eq. (38) (with s=0) becomes

$$\hat{\tau}\times\vec{E}=\upsilon\int \vec{B}_0(k)A(\vec{r},t)\,dk=\upsilon\vec{B}$$

and, in real form, it reads $\hat{\tau}\times\mathrm{Re}\vec{E}=\upsilon\,\mathrm{Re}\vec{B}$. Geometrically, this means that the $(\mathrm{Re}\,\vec{E}, \mathrm{Re}\,\vec{B}, \hat{\tau})$ define a right-handed rectangular system.

3. As shown in [7,8], the $\vec{E}$ and $\vec{B}$ are always mutually perpendicular in a *monochromatic* e/m wave of definite frequency ω, traveling in a conducting medium. Such a wave is represented in real form by the equations

$$\vec{E}(\vec{r},t)=\vec{E}_0\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\alpha)\ ,$$
$$\vec{B}(\vec{r},t)=\frac{\sqrt{k^2+s^2}}{\omega}(\hat{\tau}\times\vec{E}_0)\,e^{-s\hat{\tau}\cdot\vec{r}}\cos(k\hat{\tau}\cdot\vec{r}-\omega t+\beta)$$

where $\vec{E}_0$ is a real vector and where $\tan(\beta-\alpha)=s/k$. This perpendicularity between $\vec{E}$ and $\vec{B}$ ceases to exist, however, in a non-monochromatic wave of the form (28).

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