A note on the principle of superposition in electrodynamics

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In textbooks in electromagnetism the principle of superposition is usually referred to in the context of electrostatics and is justified by Coulomb's law and by the superposition principle for forces postulated in classical mechanics. At a deeper level of analysis, the superposition principle for time-dependent electromagnetic fields is a direct consequence of the linearity of Maxwell's system of equations. The analogous principle for forces is a separate axiom in mechanics, independent of Newton's laws.

In textbooks in electromagnetism, both of intermediate [1-5] and advanced [6-8] level, the principle of superposition is usually referred to in the chapter on electrostatics. The idea is very simple: As experiment shows, the interaction of any two charges is unaffected by the presence of other charges. Thus, by Coulomb's law and by the superposition principle for forces postulated in classical mechanics [9] the electric field created by a system of charges equals the vector sum of the fields due to each charge separately.

Indeed, let $\{q_k\}$ (k=1,2,...) be a set of stationary[1] charges and let $\{\vec{E}_k(\vec{r})\}$ (k=1,2,...) be the corresponding electrostatic fields created separately by each of these charges. We consider a test charge q_0 (not belonging to the set $\{q_k\}$) placed at some point $\vec{r}$ of space and we call $\vec{F}_k$ the force on q_0 due to the field $\vec{E}_k$ created by q_k. By the superposition principle for forces, the total force on q_0 by the electric field of the entire system $\{q_k\}$ is the vector sum $\vec{F}=\sum_i \vec{F}_i$. Consider now a vector field whose value at the location of q_0 is

$$\vec{E}(\vec{r})=\frac{\vec{F}}{q_0}=\sum_i \frac{\vec{F}_i}{q_0} \; .$$

By Coulomb's law, the force on q_0 due to q_i is proportional to q_0, so that the quotient $\vec{F}_i/q_0$ is independent of q_0 and uniquely defines the electric field $\vec{E}_i$ due to q_i at the location of q_0. Hence, the vector sum

$$\vec{E}(\vec{r})=\sum_i \vec{E}_i(\vec{r})$$

is independent of the test charge q_0 and represents the electric field produced by the entire collection of charges $\{q_k\}$.

[1] Relative to an inertial observer [9].

Notice that the above proof rests critically on two assumptions: (*a*) the force exerted by a charge q_k on q_0 is independent of the forces exerted on q_0 by other charges; (*b*) Coulomb's law is valid. As mentioned above, assumption (*a*) is related to the principle of superposition for forces[2] (one might call it "Newton's fourth law"); namely, the total force on a particle due to its simultaneous interaction with several objects is equal to the vector sum of the forces due to each object acting independently on the particle. As for Coulomb's law, it is the physical content of Gauss' law for the electric field, the latter law constituting the first of Maxwell's equations for the electromagnetic (e/m) field. It is thus an interesting exercise to check that the Maxwell system of equations is consistent with the principle of superposition in its most general form.

The Maxwell equations for the e/m field $(\vec{E},\vec{B})$ is a system of linear first-order partial differential equations:

$$\begin{aligned} &\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} && \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \\ &\vec{\nabla}\cdot\vec{B}=0 && \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

where the charge and current densities $\left(\rho(\vec{r},t),\,\vec{J}(\vec{r},t)\right)$ are subject to the equation of continuity

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \tag{2}$$

required for charge conservation.

Consider a region Ω of space and let $\left(\rho_k(\vec{r},t),\,\vec{J}_k(\vec{r},t)\right)$ (k=1,2,...) be a collection of charge and current distributions within Ω. Each pair $(\rho_k,\vec{J}_k)$ is subject to the condition

$$\vec{\nabla}\cdot\vec{J}_k+\frac{\partial\rho_k}{\partial t}=0 \tag{3}$$

We assume that there are no charges and/or currents in the exterior of Ω, so that the e/m field in Ω is *due exclusively to the sources contained in* Ω. Each individual distribution $(\rho_k,\vec{J}_k)$ will give rise to a corresponding e/m field $(\vec{E}_k,\vec{B}_k)$ satisfying the Maxwell system:

$$\begin{aligned} &\vec{\nabla}\cdot\vec{E}_k=\frac{\rho_k}{\varepsilon_0} && \vec{\nabla}\times\vec{E}_k=-\frac{\partial\vec{B}_k}{\partial t} \\ &\vec{\nabla}\cdot\vec{B}_k=0 && \vec{\nabla}\times\vec{B}_k=\mu_0\vec{J}_k+\varepsilon_0\mu_0\frac{\partial\vec{E}_k}{\partial t} \end{aligned} \tag{4}$$

[2] First stated by Daniel Bernoulli after Newton's death.

We now define a total distribution $(\rho, \vec{J})$ in *Ω* by

$$\rho(\vec{r},t)=\sum_i \rho_i(\vec{r},t)\ , \quad \vec{J}(\vec{r},t)=\sum_i \vec{J}_i(\vec{r},t) \tag{5}$$

By using (3) and by taking into account the linearity of the *div* and ∂/∂*t* operators, the continuity equation (2) may easily be verified for the total distribution (5). We also define the pair of vector functions $(\vec{E}, \vec{B})$ in *Ω* by

$$\vec{E}(\vec{r},t)=\sum_i \vec{E}_i(\vec{r},t)\ , \quad \vec{B}(\vec{r},t)=\sum_i \vec{B}_i(\vec{r},t) \tag{6}$$

where $(\vec{E}_k, \vec{B}_k)$ is the e/m field produced by the distribution $(\rho_k, \vec{J}_k)$. We propose to show that $(\vec{E}, \vec{B})$ is the e/m field in *Ω* produced by the total distribution $(\rho, \vec{J})$. For this to be the case it is sufficient that the pair $(\vec{E}, \vec{B})$ satisfy the Maxwell system (1) for the distribution $(\rho, \vec{J})$, given that, by assumption, there are no sources outside *Ω* that might contribute to the e/m field inside *Ω*.

By substituting the sums (6) for the vector functions $(\vec{E}, \vec{B})$ into Maxwell's equations (1) and by taking relations (4) and (5) into account, it is not hard to show that the system (1) is indeed satisfied. For example,

$$\begin{aligned}\vec{\nabla}\times\vec{B} &= \vec{\nabla}\times\sum_i \vec{B}_i = \sum_i \vec{\nabla}\times\vec{B}_i \overset{(4)}{=} \sum_i \left(\mu_0 \vec{J}_i + \varepsilon_0 \mu_0 \frac{\partial \vec{E}_i}{\partial t}\right) \\ &= \mu_0 \sum_i \vec{J}_i + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\sum_i \vec{E}_i = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\end{aligned}$$

We conclude that

> if the distributions $(\rho_k, \vec{J}_k)$ independently produce the corresponding e/m fields $(\vec{E}_k, \vec{B}_k)$ (k=1,2,...) in a region *Ω*, then the e/m field in *Ω* produced by the total distribution (5) is given by the vector sums in (6).

Notice that this generalized form of the superposition principle for time-dependent e/m fields rests on the linearity of Maxwell's differential equations. Thus, in electromagnetism the principle of superposition is "built into" the fundamental equations of the theory from the outset, which is not the case with Newtonian mechanics where the analogous principle for forces must be added *a posteriori* to the system of basic laws (see, e.g., [9]).

References

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[3] https://nausivios.snd.edu.gr/docs/2012C2.pdf ; new version: https://arxiv.org/abs/1205.2326