The many faces of the exponential function

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The real exponential function is defined as the inverse of the logarithmic function. Representations of the exponential function both as the limit of an infinite sequence and as an infinite series are given. The linear independence of the set $\{\exp(kx)\}$ for any number of real values of k is proven.

Problem: Let a be a positive real number. We know how to define $a^{m/n}$ with m and n integers. But, how do we define a^x for a general, real x that may be an *irrational* number, i.e., cannot be written as a quotient of integers m and n?

Well, if it is difficult to define a function directly, we may try defining the inverse function (assuming it exists). To this end, we consider the function

$$\ln x = \int_1^x \frac{1}{t}\, dt\ , \quad x > 0 \tag{1}$$

Then,

$$(\ln x)' = 1/x$$

where the prime denotes differentiation with respect to x. Note in particular that $\ln 1=0$. It can also be shown [1] that, for $a, b \in R^+$, $\ln(ab)=\ln a+\ln b$, $\ln(a/b)=\ln a-\ln b$. Thus, $\ln x$ is a logarithmic function in the usual sense.

The function $\ln x$ is increasing for $x>0$ (indeed, its derivative $1/x$ is positive for $x>0$). Since $\ln x$ is monotone, this function is invertible. Call $\exp x$ the inverse of $\ln x$. That is,

$$y = \exp x \quad \Leftrightarrow \quad x = \ln y\ .$$

This means that

$$\exp(\ln y) = y \quad \text{and} \quad \ln(\exp x) = x\ .$$

It can be shown [1] that $\exp x$ is an exponential function in the usual sense, i.e., it has the form $\exp x = e^x$ for some real constant $e>0$, to be determined. We write

$$y = e^x \quad \Leftrightarrow \quad x = \ln y \quad (x \in R\ ,\ y \in R^+)$$

so that

$$e^{\ln y} = y \quad \text{and} \quad \ln(e^x) = x\ .$$

Note in particular that, for $x=0$ we have $e^0=1$ and $\ln 1=0$, as required. Also, for $x=1$ we have that $\ln(e^1) = 1$ and, by the definition (1) of the logarithmic function,

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1\ .$$

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We will now show that the function e^x ($x \in R$) can be expressed as the limit of a certain infinite sequence:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad (x \in R) \tag{2}$$

Then, for any $a \in R^+$ we will have that $a = e^{\ln a} \Rightarrow$

$$a^x = e^{x \ln a} = \lim_{n\to\infty}\left(1+\frac{x\ln a}{n}\right)^n .$$

Proposition 1. Given a function $u = f(x)$ that assumes positive values for all x in its domain of definition, the derivative of $\ln[f(x)]$ is given by

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} \tag{3}$$

Proof. $\frac{d}{dx}\ln f(x) = \frac{d(\ln u)}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{f'(x)}{f(x)}$.

Proposition 2. The derivative of e^x is given by $(e^x)' = e^x$.

Proof. $\ln(e^x) = x \Rightarrow [\ln(e^x)]' = 1 \Rightarrow (e^x)'/e^x = 1 \Rightarrow (e^x)' = e^x$, where we have used relation (3) for the derivative of $\ln(e^x)$.

Corollary: $[\exp f(x)]' = f'(x)\exp f(x)$.

Now, consider the function $g(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ ($x \in R$). We have:

$$g'(x) = \lim_{n\to\infty}\left[n\left(1+\frac{x}{n}\right)^{n-1}\frac{1}{n}\right] = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^n\left(1+\frac{x}{n}\right)^{-1}\right]$$

$$= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-1} = g(x)\cdot 1 = g(x).$$

Moreover, $g(0)=1$. Hence the function $y=g(x)$ satisfies the differential equation $y'=y$ with initial condition $y=1$ for $x=0$. On the other hand, the function $y=e^x$ satisfies the same differential equation with the same initial condition. Since the solution of this differential equation with given initial condition is unique, we conclude that the functions $g(x)$ and e^x must be identical. Therefore relation (2) must be true.

We note that, for $x=1$, Eq. (2) gives

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad (\simeq 2.72) \tag{4}$$

This is the formula by which the number e is usually defined.

In the same spirit we may show that another possible representation of the exponential function e^x is in the form of a power (Maclaurin) series [2]:

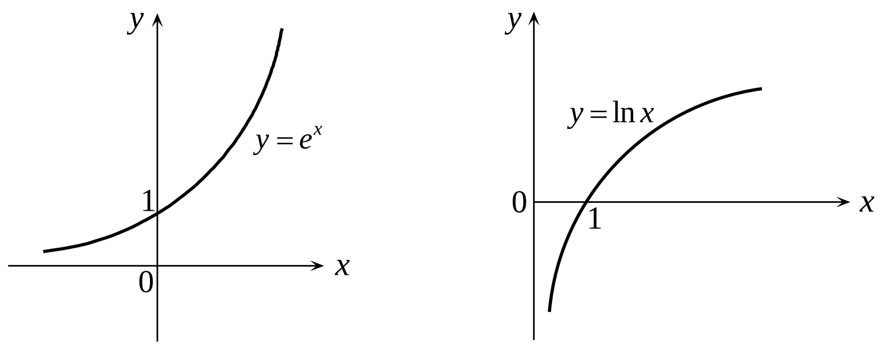
$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \quad (x\in R) \tag{5}$$

Indeed, notice that the x-derivative of this series is the series itself, as well as that the value of the series is equal to 1 for x=0. Although expressions (2) and (5) do not look alike, they represent the *same* function, exp x ! (*Note:* Two functions of x are considered identical if they have the same domain D of definition and assume equal values for all $x\in D$.)

We defined a^x (a>0, $x\in R$) in a rather indirect way by first defining the function e^x as the inverse of the function $\ln x$ and then by writing $a^x=e^{x\ln a}$. There is, however, a more direct definition of a^x. Let $x_1, x_2, \ldots, x_n, \ldots$ be *any* infinite sequence of *rational* numbers x_n such that $\lim_{n\to\infty} x_n = x\in R$. [*Question:* Can a sequence of rational numbers have an *irrational* limit? Yes! See, e.g., the expression (4) for e, where the latter number *is* irrational (see, e.g., [3]).] We now define a^x as follows:

$$a^x=\lim_{n\to\infty} a^{x_n} \quad (a>0,\ x\in R).$$

Since x_n is a rational number for all n, raising a to a rational number should not be a problem. Note that the value of a^x does not depend on the specific choice of the sequence x_n, as long as the limit of this sequence is x.



Graphs of exponential and logarithmic functions.

Theorem. Consider the function $F(x)=\sum_{i=1}^{n} A_i \exp(k_i x)$, where the real constants k_i are different from each other. If $F(x)\equiv 0$ for all x, then A_i =0 for all i=1,2,...,n. Thus, the functions $\{\exp(k_i x),\ i=1,2,\ldots,n\}$ are a linearly independent set.

Proof. We will prove the theorem by induction. The case n=1 is obvious, given that the function exp(kx) is nonzero for any finite x. Let us check the case n=2. Thus, assume that

$$F(x) = A_1 \exp(k_1 x) + A_2 \exp(k_2 x) \equiv 0 \quad (\text{for all real } x)\ .$$

Since $F(x)$ is the constant function, its derivative must vanish identically:

$$F'(x) = k_1 A_1 \exp(k_1 x) + k_2 A_2 \exp(k_2 x) \equiv 0\ .$$

Then, $F'(x) - k_1 F(x) = 0 \Rightarrow (k_2 - k_1) A_2 \exp(k_2 x) \equiv 0 \Rightarrow A_2 = 0$, given that, by assumption, $k_2 \neq k_1$. Thus, $F(x) = A_1 \exp(k_1 x) \equiv 0 \Rightarrow A_1 = 0$. For $n=3$, let

$$F(x) = A_1 \exp(k_1 x) + A_2 \exp(k_2 x) + A_3 \exp(k_3 x) \equiv 0\ .$$

Then, $F'(x) - k_1 F(x) = 0 \Rightarrow (k_2 - k_1) A_2 \exp(k_2 x) + (k_3 - k_1) A_3 \exp(k_3 x) \equiv 0 \Rightarrow A_2 = A_3 = 0$ (case $n=2$). Hence, $F(x) = A_1 \exp(k_1 x) \equiv 0 \Rightarrow A_1 = 0$. Now, assume that the theorem is valid for some value of $n > 2$. We want to show that it is also valid for $n+1$. To this end, we consider the function $F(x) = \sum_{i=1}^{n+1} A_i \exp(k_i x)$. It is convenient to rename the $(n+1)$-term as 0-term, and write

$$F(x) = A_0 \exp(k_0 x) + \sum_{i=1}^{n} A_i \exp(k_i x) \equiv 0$$

so that $F'(x) = k_0 A_0 \exp(k_0 x) + \sum_{i=1}^{n} k_i A_i \exp(k_i x) \equiv 0$. Then,

$$F'(x) - k_0 F(x) = 0 \ \Rightarrow\ \sum_{i=1}^{n} (k_i - k_0) A_i \exp(k_i x) \equiv 0 \ \Rightarrow\ A_1 = A_2 = \cdots = A_n = 0$$

given that, by assumption, $k_i \neq k_0$, as well as that the theorem is assumed to be valid for a sum with n terms. Thus, $F(x) = A_0 \exp(k_0 x) \equiv 0 \Rightarrow A_0 = 0$. In conclusion:

> *The functions* $\{\exp(k_i x),\ i = 1,2,...\}$ *form a linearly independent set for different values of the real constants* k_i .

References

[1] D. D. Berkey, *Calculus*, 2nd Edition (Saunders College, 1988), Chap. 8.

[2] C. J. Papachristou, *Elements of Mathematical Analysis: An Informal Introduction for Physics and Engineering Students* (Springer, 2024).

[3] https://mindyourdecisions.com/blog/2015/06/18/lets-prove-e-2-718-is-irrational-3-methods/